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Dynamics of decelerating pulses at a dielectric layer

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Abstract Time dependent electromagnetic pulses propagating with deceleration along the dominant propagation direction, and passing through a dielectric layer are investigated in the 1D + T case. This is achieved using an integral equation in the paraxial approximation obtained via the Green's function symmetric with respect to the propagation direction.

Keywords Integral equation · Green's function · Paraxial approximation · Electromagnetic pulse · Dielectric layer

1 Introduction

In the overwhelming majority of investigations accelerated Airy beams (Siviloglou and Christodoulides 2007) have harmonic temporal dependence and are normally considered in a homogeneous medium. Intensive theoretical and experimental investigations of Airy beams are motivated by their unusual features, the most interesting of which is accelerating motion. Publications with theoretical formulation and experimental confirmation were followed by many works on the properties of Airy beams that have generated very interesting applications, some of which are already realised. Time dependent Airy pulses

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have very complex behaviour in temporal domain (Kaganovsky and Heyman 2011), and this behaviour can change significantly when a pulse propagates in an inhomogeneous medium. The main features of such changes can be revealed in a simple model problem about pulse reflection from, and transmission through, a dielectric layer. The great importance of the interaction between accelerating beams and a periodic diffraction grating has been recently shown in (Lumer et al. 2015) where self-imaging (the Airy-Talbot effect) along curved trajectories has been considered. The problem of pulse interaction with the layer can be solved by many methods, but it seems convenient to use integral methods, describing the problem as a whole and taking into account initial and boundary conditions, that facilitate studying the pulse dynamics in such phenomena. This study can be organised around two complementary approaches, progressing and oscillatory, that are closely related to local versus global descriptions (Felsen and Whitman 1970). The progressing description operates with wave fronts, and it represents point-to-point propagation during which the wave fronts interact with the physical environment locally along their trajectories. The oscillatory approach operates with modes or resonances which form standing waves over extended, often global, portions of the physical environment. These two approaches reveal the double nature of the oscillation and wave processes in the waveguide structures. A time-spatial approach has shown an important physical difference between spatial and temporal acceleration (Kaminer et al. 2011). An approach which considers the problem when electromagnetic pulses are generated by an external source is given without making any assumptions on the temporal dependence of the field but with the assumption of the paraxial approximation (Nerukh et al. 2012; Nerukh and Nerukh 2013). It provides a physically natural picture of the phenomenon, avoids the problem of backward flow of time, and has a clear physical meaning by being free of exotic corollaries.

Here, the problem is described via an integral equation derived, under a paraxial approximation, using the Green's function which is symmetric with respect to the propagation direction. A time dependent electromagnetic Airy pulse generated by an extrinsic current is considered. The dynamics of this pulse when passing a dielectric layer is investigated in 1D + T case. The analytical expressions for the reflected and transmitted waves are derived and a graphical illustration is given.

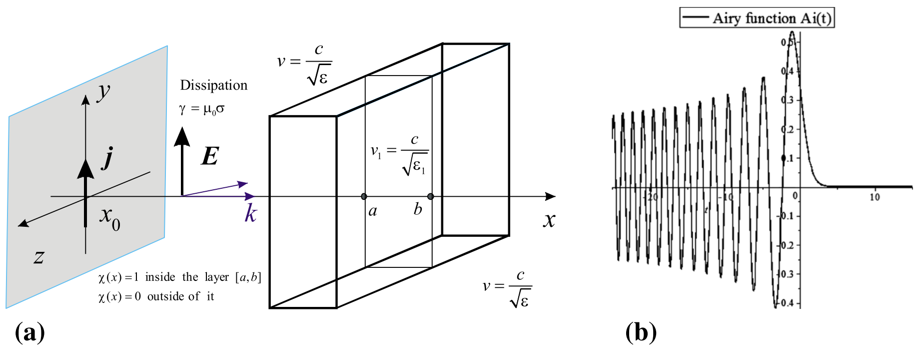

Fig. 1 **a** A plate source current ***j*** radiates a time-spatial pulse which propagates along the direction perpendicular to the current plane, reflects from and transmits through a dielectric layer (at the *left-hand side*). **b** The current in the form of the Airy function is shown at the *right-hand side* as an example

2 The statement of the problem

We consider the dynamics of an electromagnetic pulse radiated by an extrinsic electric current $j(t, x)$ in the plane $x = x_0$ perpendicular to the direction of the radiation propagation and located in a dissipative nondispersive medium with permittivity ε, permeability μ, and conductivity σ, Fig. 1a. In this case the radiated electric field has the same orientation as the current and in the case of an inhomogeneous medium it is controlled by the inhomogeneous wave equation with the right-hand side determined by the source current. If a medium has an inhomogeneity represented by a layer $[a, b]$ with the permittivity ε_1 and conductivity σ_1 it can be accounted for by a term in the right-hand side as follows:

$$\frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} + \gamma\frac{\partial E}{\partial t} - \frac{\partial^2 E}{\partial x^2} = \left(\frac{1}{v^2} - \frac{1}{v_1^2}\right)\chi(t,x)\frac{\partial^2 E}{\partial t^2} + (\gamma - \gamma_1)\chi(t,x)\frac{\partial E}{\partial t} - \mu_0\mu\frac{\partial j}{\partial t} \qquad (1)$$

Here, $v^2 = 1/(\varepsilon_0\varepsilon\mu_0\mu)$, $v_1^2 = 1/(\varepsilon_0\varepsilon_1\mu_0\mu)$ ε_0 and μ_0 are the permittivity and permeability of vacuum respectively, the dissipation is designated by $\gamma = \mu_0\mu\sigma$ and $\gamma_1 = \mu_0\mu\sigma_1$ and the characteristic function $\chi(t, x) = 1$ inside the layer $[a, b]$ and $\chi(t, x) = 0$ outside of it in this equation.

We consider the problem when the pulse propagates dominantly along the x axis, $E(t, x) = F(t, x)e^{-ik|x|}$, where $k > 0$, and the paraxial approximation $\left|2ik\frac{\partial F}{\partial x}\right| \gg \left|\frac{\partial^2 F}{\partial x^2}\right|$ for an envelope $F(t, x)$ is fulfilled. Omitting the second derivative in (1) gives the equation for the envelope

$$\begin{aligned}&\frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} + \gamma\frac{\partial F}{\partial t} + 2ik\,\mathrm{sign}(x)\frac{\partial F(t,x)}{\partial x} + 2ik\delta(x)F(t,x) + k^2F(t,x)\\ &= \frac{v_1^2 - v^2}{v^2v_1^2}\chi(t,x)\frac{\partial^2 F}{\partial t^2} + (\gamma - \gamma_1)\chi(t,x)\frac{\partial F}{\partial t} - \mu_0\mu\frac{\partial j}{\partial t}e^{ik|x|}\end{aligned} \qquad (2)$$

The Green's function of this equation obeying the Eq. (3) following from (2)

$$\frac{1}{v^2}\frac{\partial^2 G}{\partial t^2} + \gamma\frac{\partial G}{\partial t} + 2ik\,\mathrm{sign}(x)\frac{\partial G(t,x)}{\partial x} + 2ik\delta(x)G(t,x) + k^2G(t,x) = \delta(t)\delta(x) \qquad (3)$$

has the form

$$G(t,x) = -\frac{(1+i)v}{4\sqrt{\pi k|x|}}e^{i\left[k\frac{|x|}{2} + \frac{v^2(2kt + i\gamma|x|)^2}{8k|x|}\right]} = -\frac{i}{4\pi k}e^{ik\frac{|x|}{2}}\int_{-\infty}^{\infty} e^{-i\frac{\omega^2 - i\omega v^2\gamma}{v^2k}\frac{|x|}{2}}e^{i\omega t}d\omega \qquad (4)$$

By virtue of this function the Eq. (2) is transformed to the integral equation for the field envelope

$$\begin{aligned}F(t,x) = &F_0(t,x)\\ &-\frac{(1+i)v}{4\sqrt{\pi k}}\int_a^b dx'\int_{-\infty}^{\infty} dt'\frac{1}{\sqrt{|x-x'|}}e^{\frac{ik}{2}\left[|x-x'| + \frac{v^2\left(2k(t-t') + i\gamma|x-x'|\right)^2}{4k^2|x-x'|}\right]}\left[\frac{v_1^2 - v^2}{v^2v_1^2}\frac{\partial^2 F}{\partial t'^2} + (\gamma - \gamma_1)\frac{\partial F}{\partial t'}\right]\end{aligned} \qquad (5)$$

with the free term determined by the source

$$F_0 = \frac{(1+i)v\mu_0\mu}{4\sqrt{\pi k}} \int\limits_{-\infty}^{\infty} dx' \int\limits_{-\infty}^{\infty} dt' \frac{1}{\sqrt{|x-x'|}} e^{i\frac{k}{2}\left[|x-x'|+\frac{v^2\left(2k(t-t')+i\gamma|x-x'|\right)^2}{4k^2|x-x'|}\right]} \frac{\partial j}{\partial t'} \tag{6}$$

If the field is considered inside the layer $x \in [a, b]$ then the expression (5) is the equation for this field, $F_{in}(t, x)$. The field outside the layer, $F_{R,T}$, is found by integrating the inner field over the layer. It is easy to show that the fields satisfy the condition of continuity at layer borders, $F_R(t, a-0) = F_{in}(t, a+0)$, $F_{in}(t, b-0) = F_T(t, b+0)$.

3 The method

Converting to the Fourier transform $f(\omega, x) = \int\limits_{-\infty}^{\infty} F(t, x)e^{-i\omega t}dt$ allows to one to obtain the integral equation for the field spectra. The spectrum for the inner field, the field inside the layer, satisfies the integral equation

$$\begin{aligned} f_{in}(\omega, x) = j_0(\omega)e^{ik\Omega\frac{x}{2}} + \frac{i}{2k}e^{ik\Omega\frac{x}{2}} \int\limits_a^x dx' e^{-ik\Omega(\omega)\frac{x'}{2}} \{\omega^2 \frac{v_1^2 - v^2}{v^2 v_1^2} - i\omega(\gamma - \gamma_1)\} f_{in}(\omega, x') \\ + \frac{i}{2k} e^{-ik\Omega\frac{x}{2}} \int\limits_x^b dx' e^{ik\Omega(\omega)\frac{x'}{2}} \left\{\omega^2 \frac{v_1^2 - v^2}{v^2 v_1^2} - i\omega(\gamma - \gamma_1)\right\} f_{in}(\omega, x') \end{aligned} \tag{7}$$

where $\Omega(\omega) = \Omega' + i\Omega'' = 1 - \frac{\omega^2}{k^2v^2} + i\frac{\omega v^2 \gamma}{k^2 v^2}$. The free term in this equation follows from (6) and in the case when the whole consideration is made to the right of the source, $x - x_0 > 0$ and $x_0 < a$, it is equal to

$$F_0(t, x) = \int\limits_{-\infty}^{\infty} j_0(\omega)e^{i\left(k - \frac{\omega^2 - i\omega v^2\gamma}{v^2 k}\right)\frac{x}{2}} e^{i\omega t} d\omega \tag{8}$$

where

$$j_0(\omega) = -\frac{\mu_0\mu}{4\pi k} e^{ik\left(|x_0| - \frac{x_0}{2}\right) + i\frac{\omega^2 - i\omega v^2\gamma}{v^2 k}\frac{x_0}{2}} \omega\Phi(\omega), \tag{9}$$

where the source current is given by a spectrum $\Phi(\omega)$, $j(t, x) = \delta(x - x_0)\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \Phi(\omega)e^{i\omega t}d\omega$. The Airy pulse considered here is generated by the source current changing as the Airy function, Fig. 1b,

$$j = \delta(x - x_0)\text{Ai}(t/T)e^{\alpha t/T}/T \tag{10}$$

that has the spectrum $\Phi(\omega) = \exp(i(\omega T + i\alpha)^3/3)$ in (9).

The spectra for the field outside the layer are expressed through the inner spectrum: to the left of the layer (the reflected pulse)

$$f_R(\omega,x) = j_0(\omega)e^{ik\Omega\frac{x}{2}} + \frac{i}{2k}e^{-ik\Omega\frac{x}{2}}\int_a^b dx' e^{ik\Omega\frac{x'}{2}}\left\{\omega^2\frac{v_1^2 - v^2}{v^2v_1^2} - i\omega(\gamma-\gamma_1)\right\}f_{in}(\omega,x') \quad (11)$$

To the right of the layer (transmitted pulse) it is

$$f_T(\omega,x) = j_0(\omega)e^{ik\Omega\frac{x}{2}} + \frac{i}{2k}e^{ik\Omega\frac{x}{2}}\int_a^b dx' e^{-ik\Omega\frac{x'}{2}}\left\{\omega^2\frac{v_1^2 - v^2}{v^2v_1^2} - i\omega(\gamma-\gamma_1)\right\}f_{in}(\omega,x') \quad (12)$$

It is easy to see that the spectra also satisfy the continuity conditions at layer borders, $f_R(\omega,\, a-0) = f_{in}(\omega,\, a+0), f_{in}(\omega,\, b-0) = f_T(\omega,\, b+0)$.

Assuming the form for the inner field $f_{in}(\omega,\, x) = C_1(\omega)e^{is(\omega)x} + C_2(\omega)e^{-is(\omega)x}$ we find from (7) the equations for the wavenumber s and the coefficients $C_{1,2}$. The wavenumber s satisfying the dispersion equation

$$s^2 - k^2\Omega^2/4 - \frac{1}{2}\left[\omega^2\frac{v_1^2 - v^2}{v^2v_1^2} - i\omega(\gamma-\gamma_1)\right]\Omega = 0 \quad (13)$$

is given by the formula

$$s = \pm\frac{1}{2}k\sqrt{\Omega(2\Omega_1 - \Omega)} \quad (14)$$

Here, $\Omega_1(\omega) = \Omega_1' + i\Omega_1'' = 1 - \frac{\omega^2}{k^2v_1^2} + i\frac{\omega\gamma_1}{k^2}$ and a branch for the radical in (14) is chosen such that $\mathrm{Im}\sqrt{\Omega(2\Omega_1 - \Omega)} > 0$. The coefficients in the expression for the inner field are given by the expressions

$$C_1 = j_0(\omega)\frac{2k}{\left\{\omega^2\frac{v_1^2-v^2}{v^2v_1^2} - i\omega(\gamma-\gamma_1)\right\}}\frac{[s-k\Omega/2][s+k\Omega/2]^2}{[s+k\Omega/2]^2e^{i2sa} - [s-k\Omega/2]^2e^{i2sb}}e^{isa+ik\Omega a/2} \quad (15)$$

$$C_2 = C_1\frac{[s-k\Omega/2]}{[s+k\Omega/2]}e^{i2bs} \quad (16)$$

Substitution $f_{in}(\omega,\, x)$ to (11) and (11) gives the spectra for the reflected and transmitted pulses

$$f_R(\omega,x) = j_0(\omega)e^{ik\Omega x/2} + j_0(\omega)e^{ik\Omega(a-x/2)}\left[e^{ik\sqrt{\Omega(2\Omega_1-\Omega)}(b-a)} - 1\right]\frac{\Omega_1 - \Omega}{\Omega_1 + \sqrt{\Omega(2\Omega_1-\Omega)}}D \quad (17)$$

$$f_T(\omega,x) = 2j_0(\omega)e^{ik\Omega x/2}e^{i\left(\sqrt{\Omega(2\Omega_1-\Omega)}-\Omega\right)k\frac{(b-a)}{2}}\frac{\sqrt{\Omega(2\Omega_1-\Omega)}}{\Omega_1+\sqrt{\Omega(2\Omega_1-\Omega)}}D, \quad (18)$$

where the factor D is determined by the fraction

$$D = \frac{1}{1 - \frac{\Omega_1 - \sqrt{\Omega(2\Omega_1-\Omega)}}{\Omega_1+\sqrt{\Omega(2\Omega_1-\Omega)}}e^{ik\sqrt{\Omega(2\Omega_1-\Omega)}(b-a)}}. \quad (19)$$

4 Analysis

The wavenumber (14) turns to zero in the trivial case $\Omega = 0$ and in the case $\Omega_1 = \Omega/2$. In both cases there is uncertainty in the kernels for the reflectance (17) and the transmittance (18), but it is easy to show that the limits for these kernels exist. For example the limit for the kernel

$$K = e^{ik\Omega a}\frac{[\Omega_1 - \Omega][e^{ik\sqrt{\Omega(2\Omega_1-\Omega)}a} - 1]}{[\Omega_1 + \sqrt{\Omega(2\Omega_1 - \Omega)}]}D \tag{20}$$

exists when $\Omega_1 \to \Omega/2$ and is equal to

$$\lim_{\Omega_1\to\Omega/2} K = \frac{\Omega ka[\Omega k(b-a) + 4i]e^{ik\Omega a}}{16 + \Omega^2k^2(b-a)^2} \tag{21}$$

The wavenumber s in (14) becomes imaginary if $\Omega(2\Omega_1 - \Omega) < 0$, which is met in the hatched region in Fig. 2.

If $\Omega(2\Omega_1 - \Omega) > 0$ then the absolute value of the fraction in (19) is less than one and this function can be expanded in the series

$$D = \sum_{n=0}^{\infty}\left[\frac{\Omega_1 - \sqrt{\Omega(2\Omega_1 - \Omega)}}{\Omega_1 + \sqrt{\Omega(2\Omega_1 - \Omega)}}\right]^n e^{ink\sqrt{\Omega(2\Omega_1-\Omega)}(b-a)} \tag{22}$$

The denominators in the series terms become equal to zero only in the trivial case $v_1 = v$ but in this case there is no layer and there is no diffraction. If $\Omega(2\Omega_1 - \Omega) < 0$ then the radicals in (19) are imaginary and $\frac{\Omega_1-\sqrt{\Omega(2\Omega_1-\Omega)}}{\Omega_1+\sqrt{\Omega(2\Omega_1-\Omega)}} = e^{i\varphi}$ with the absolute value equal to one, $\left|\frac{\Omega_1-\sqrt{\Omega(2\Omega_1-\Omega)}}{\Omega_1+\sqrt{\Omega(2\Omega_1-\Omega)}}\right| = 1$. In this case the series (22) consists of evanescent terms

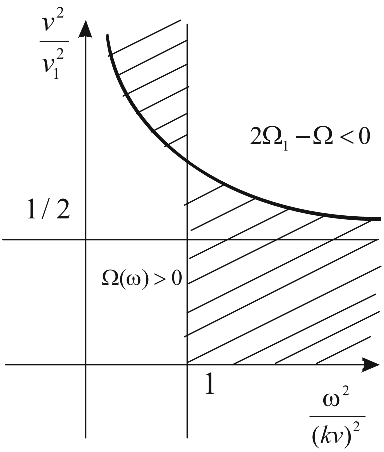

Fig. 2 The region hatched on the diagram for the relation between the frequency and the ratio of refractive indices $n^2 = c/v$, $n_1^2 = c/v_1$ corresponds to the imaginary wavenumber, $\Omega(2\Omega_1 - \Omega) < 0$

$$D = \sum_{n=0}^{\infty} e^{in\varphi - nk\left|\sqrt{\Omega(2\Omega_1 - \Omega)}\right|(b-a)} \tag{23}$$

where φ is the argument of the fraction. In both cases the terms in the series (22) and (23) decrease as power functions.

5 Results

The inverse Fourier transform of (17) and (18) gives the reflected and transmitted pulses

$$F_R(t,x) = \int_{-\infty}^{\infty} j_0(\omega) e^{ik\Omega(a-x/2)} \left[e^{ik\sqrt{2\Omega(\Omega_1-\Omega)}(b-a)} - 1\right] \frac{\Omega_1 - \Omega}{\Omega_1 + \sqrt{2\Omega(\Omega_1 - \Omega)}} D e^{i\omega t} d\omega \tag{24}$$

$$F_R(t,x) = \int_{-\infty}^{\infty} j_0(\omega) e^{ik\Omega(a-x/2)} \left[e^{ik\sqrt{2\Omega(\Omega_1-\Omega)}(b-a)} - 1\right] \frac{\Omega_1 - \Omega}{\Omega_1 + \sqrt{2\Omega(\Omega_1 - \Omega)}} D e^{i\omega t} d\omega \tag{25}$$

The time-spatial behaviour of the pulses is calculated in dimensionless variables $\tau = tv/L$, $\xi = x/L$, $\kappa = kL$ where $L = b - a$ is the layer width.

The pictures in Fig. 3 illustrate the time-spatial behaviour of the initial and reflected pulses. Comparison of temporal and spatial changing of these pulses is shown in in detail Fig. 4.

The same detailed comparisons are made for the transmitted pulse in Fig. 5. The level curves for the transmitted pulse, Fig. 5a, are similar to the initial one but Fig. 5b seems to contain a contradiction as the transmitted pulse amplitude is greater than the initial pulse amplitude.

This contradiction is removed if one takes into account the displacement of the initial wave due to the action of the layer, Fig. 6. The displacement $\Delta\xi = 0.0565$ of the

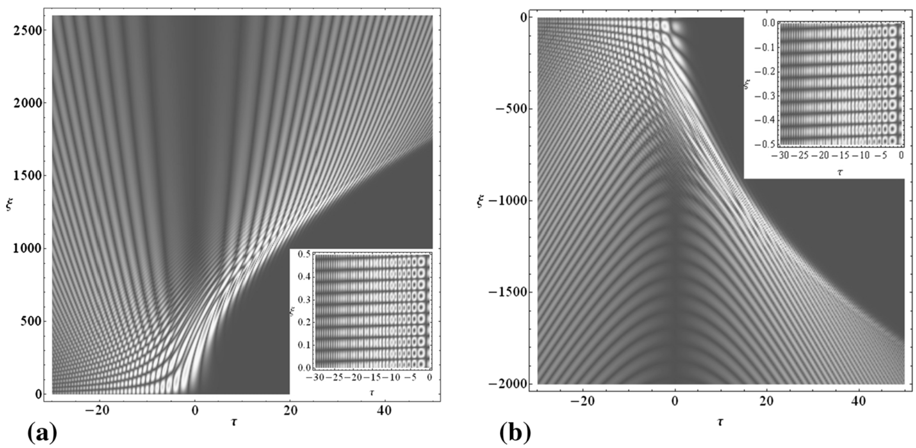

Fig. 3 The level curves for the electric fields: **a** the initial pulse $E_0(\tau, \xi)$; **b** the reflected pulse $E_{refl}(\tau, \xi)$. Details near the point $\xi = 0$ are in the *insets*

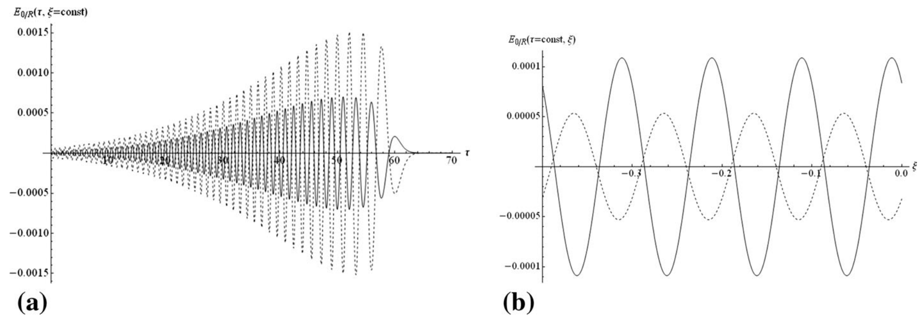

Fig. 4 Comparison of the initial pulse (*dashed line*) and the reflected pulse (*solid line*) at the layer entrance: **a** the temporal behaviour at the point $\xi = -0.1$; **b** the spatial distribution of the fields near the illuminated boundary of the layer

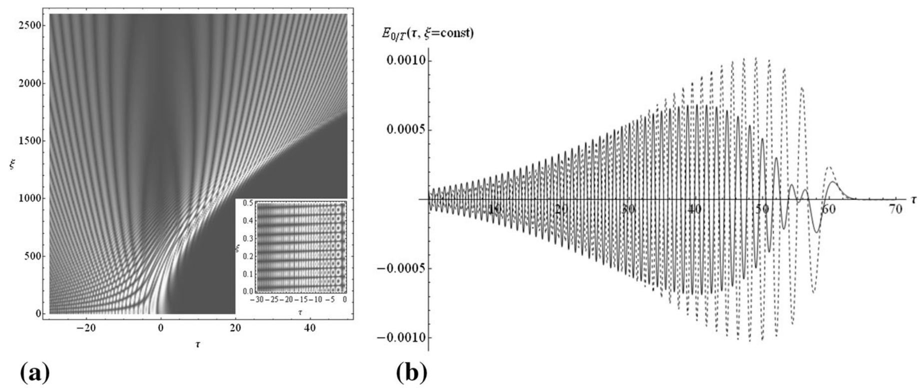

Fig. 5 **a** The level curves for the electric field of the transmitted pulse E_{trans} (τ, ξ) (details near the point $\xi = 0$ in the *inset*); **b** comparison of temporal behavior of the initial pulse (*dashed line*) and the transmitted pulse (*solid line*) at the point $\xi = 2$

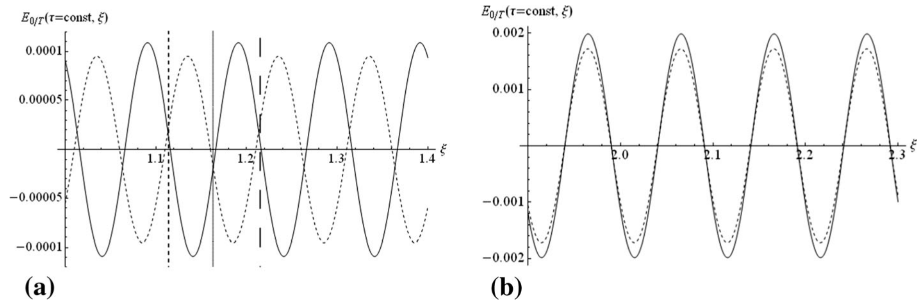

Fig. 6 **a** Comparison of spatial profiles of the initial pulse (*solid line*) and the transmitted one (*dashed line*) at the moment $\tau = 5$. **b** The same comparison if one takes into account the displacement of the initial wave that is equal to $\Delta\xi = 0.0565$

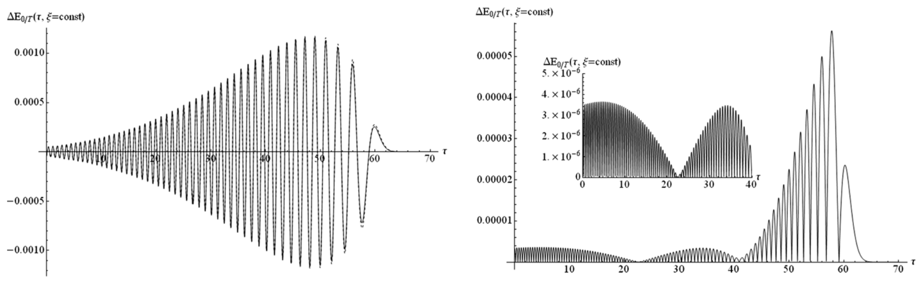

Fig. 7 **a** Comparison of the initial wave (*solid line*) and the transmitted wave (*dashed line*) at the moment $\tau = 5$ if the initial wave is displaced by $\Delta\xi = 0.0565$, and **b** the absolute value of the difference of these waves amplitudes

transmitted wave with respect to the initial one, arising from the influence of the layer, must be added to the initial wave to balance the phases of these waves.

The balanced distribution of the waves at the point $\xi = 2$ is shown in Fig. 7a, and the difference of the initial wave and the wave multiplied by the phase factor due to the presence of the layer is shown in Fig. 7b.

If such a displacement is taken into account, the difference between these amplitudes is less than 0.5 % on the interval $\xi \in [0, 40]$ and not more than 5 % over the whole interval considered. The increase of the error on the interval [40, 75] is determined by the transformation of the Airy pulse by the layer not simply displaced in time and space.

6 Conclusion

An integral equation describing the interaction of electromagnetic pulses with an obstacle in the form of a dielectric layer is derived in the paraxial approximation. The reflection and transmission of an initial Airy pulse is considered as an example. The analysis of the process is made and it is shown that these secondary pulses are decelerating like the initial one.

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