Method of Calculating Fourier Coefficients of Three Variable Functions Using Tomogram

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***Abstract*—A method for calculating Fourier coefficients of functions of three variables using tomograms is proposed and investigated. Earlier Lytvyn O.M. proposed and introduced the method of calculating Fourier coefficients of the function of two variables using integrals of this function along a given system of direct. It is believed that these integrals - projections or projection data arrive from a computer tomograph.**

The theoretical positions of the method devoted to the calculation of the Fourier coefficients of the function of two variables using projective data from the approximate function are given in the paper. A characteristic feature of the proposed method is that it is not based on the inverse transformation of Radon, which is used in computer tomography and does not need to go to polar coordinates when calculating Fourier coefficients.

In this paper, for the first time, a generalization of this method is proposed in the case of calculating Fourier coefficients of functions of three variables using tomograms placed on a system of planes parallel to a given system of planes. It is believed that these tomograms come from a computer tomograph. In the three-dimensional case studied in this paper, known integrals of the approximate function are considered in two-dimensional regions lying in planes parallel to the given base planes, which are completely determined by indexes of Fourier coefficients. The explicit formulas for the transition from the Cartesian coordinate system to another Cartesian coordinate system are given in the work, one of whose variables is determined by the indicated base plane, and the other two variables create a two-dimensional Cartesian system located on the specified base plane. To calculate the Fourier coefficients of the function of three variables for different base planes, we obtain two-dimensional integrals. The choice of the necessary base planes is proposed by replacing the rapidly expanding complex exponential factor with periodic piecewise - stable splines of the best approximation.

Keywords—3D-Fourier coefficients, tomograms, computer tomography, reconstruction, image, discontinuous spline, sum Fourier.

I. INTRODUCTION

In 1917, I. Radon proved that each function of n variables $(n \geq 2)$ can be uniquely determined not only by its values at separate points, but also by means of integrals along the system of lines or systems of planes (for $n \geq 3$) [6].

In 1979, A. Cormac and H. Hounsfield were awarded the Nobel Prize for creating a computer tomograph. Such a tomography allows, on the basis of the inverse Radon transformation, to approximate the internal structure of the body according to known projections - Radon's data.

In [3, 4, 5], the method of an approximate representation of the function of two variables by finite Fourier sums was investigated, in which the Fourier coefficients are found helping projections using the direct and inverse Radon transformation This method is known as the Direct Fourier Method (DFM). To obtain experimental data in the DFM method, a parallel scan scheme is used. The main idea of the DFM method is to find Fourier transforms of projections p and use them to find the Fourier transform of a function f that describes the investigated image. Numerical implementation includes a discrete Fourier transform p and inverse discrete function transformation f using a fast Fourier transform.

Therefore, to restore the image $N \times N$ the method DFM falls into a number of very fast methods, with the number of arithmetic operations is proportional $N^2 \log N$. But there are two problems that arise with its numerical implementation. The first problem is the need to perform an interpolation procedure in Fourier's space, which is a complicated procedure.

The authors of the DFM method assert that all polynomial interpolation methods are not suitable for this purpose. Therefore, they propose another method of interpolation, constructed using Fourier series and the central section theorem (Sampling Theorem). The second problem is that in practice contours of images have sharpening, angular points, which leads to an increase in the breaks in the function f. It is well known that in the approximation of this function by finite Fourier sums, the Gibbs phenomenon arises. It generates nonphysical oscillations in the form, that is oscillations, which the original does not have.

The authors of the DFM method claim that they have investigated several methods of combating the Gibbs phenomenon and as a result proposed two different ways to eliminate most of the oscillations. The first one is a simple application of an exponential filter for Fourier coefficients. The second way is using the fact that the amplitude of the oscillation is proportional to the jump of the function f. Jumps have different values for typical applications

Therefore, some of the projections corresponding to the skull are removed before the image is calculated.

In 2000, O. M. Lytvyn [1] proposed a method for calculating the Fourier coefficients of functions of two variables by means of projections - integrals from the approximate function along a given system of basic lines. The placement of these lines is determined by the indexes in the Fourier coefficients. To obtain experimental data-projections, this method also uses a parallel scan scheme. It is proposed to search for an approximate solution of the two-dimensional problem of Radon computed tomography in the form of finite Fourier sums. But for the purpose of finding Fourier coefficients, using the projections emanating from a computer tomograph, Radon transformation properties are not used. Only the projection definition is used as an integral along the line passing through the object of the study.

This new method for restoring two-dimensional images using projections and finite Fourier sums is more efficient than the above method from the computational point of view. But it requires a special choice of a set of basic lines along which experimental data are obtained. That is, it is proposed to use a new scan scheme.

The authors also investigated objects described by the functions of two variables with breaks of the first kind on a certain system of lines. It is proposed to use discontinuous splines of two variables to automatically find these lines of discontinuity. And also the method of O. M. Lytvyn's calculation of Fourier coefficients of functions of two variables with the help of periodic discontinuous splines of one variable and projections.

The given method was investigated in [2] and other works. The following briefly describes the main aspects and formulas of this approach for a two-dimensional case for completeness and systematic presentation of the material.

In this paper, for the first time, a generalization of this method is proposed for a three-dimensional case. It is proposed to seek a function approximation $f(x,y,z)$ in the form of finite Fourier sums. Fourier coefficients are calculated using tomograms coming from a computer tomograph. In this case, direct and inverse Radon transformation is not used. Only integrals of the approximate function of three variables in terms of flat regions lying in planes parallel to the base system of planes are known. The placement of these planes in space is completely determined by the indexes of the Fourier coefficients. Clearly written formulas for the transition from the Cartesian coordinate system $Oxyz$ to another coordinate system $Otuv$, determined by a specific base plane.

II. FORMULAS FOR CALCULATING FOURIER COEFFICIENTS OF FUNCTIONS OF TWO VARIABLES $f(x,y)$ BY MEANS OF PROJECTIONS

Calculation of Fourier coefficients using projections by O. M. Lytvyn is given in [1]. Here is the information.

The task of image reconstruction is to restore the function $f(x,y)$ according to known projection data – the values of integrals γ_μ along the lines L_μ that cross the object of the study:

$$\int_{L_\mu} f(x,y)dl = \gamma_\mu,\ \mu=\overline{1,Q},$$

In the future we will assume that the object of research belongs to the square $D=[0,1]^2$.

This problem can be interpreted as the task of restoring the function $f(x,y)$ – the absorption coefficient of X-rays. Or as a problem of studying the density $f(x,y)$ inside a body on the plane Oxy by method of X-ray computer tomography. Projection data, which is an experimental data, comes from a computer tomograph.

According to the investigated method, the problem was solved in the form of a Fourier sum.

$$f(x,y) \approx S_{N,N}(x,y) = \sum_{k=-N}^{N}\sum_{l=-N}^{N} F_{k,l} e^{i2\pi(kx+ly)}.$$

Fourier coefficients are calculated by the formula:

$$F_{k,l} = \iint_D f(x,y) e^{-i2\pi(kx+ly)} dxdy.$$

The peculiarity and advantage of the developed method is that explicit formulas have been found for the approximate calculation of Fourier coefficients of the function of two variables through the values of projections. This led to the solution of the problem to the computation of integrals. The choice of a system of direct, along which projective data is given, is due to the values of the indices k and l in the Fourier sum. Consequently, both the form of integrals and the formulas for their calculation are also due to the values of the k and l indices in the Fourier sum.

For the calculation of Fourier coefficients $F_{k,l}$ using projections, cases were considered separately for the values and signs k and l, as well as their relative positioning.

Next, for the calculation of Fourier coefficients $F_{k,l}$ by means of projections, we separately consider cases involving the signs k and l. In particular, for the case $k>0$ and $l>0$ make the replacement of variables:

$$kx+ly=t,\ -lx+ky=v,$$

$$x=x(t,v)=\frac{kt-lv}{k^2+l^2},\ y=y(t,v)=\frac{lt+kv}{k^2+l^2}.$$

As a result, the domain of integration D will split into three sub-regions, D_1, D_2, D_3 where $k > l$ either $k < l$, but to two sub-regions D_1, and D_3 if $k = l$. These cases are shown in Figure 1.

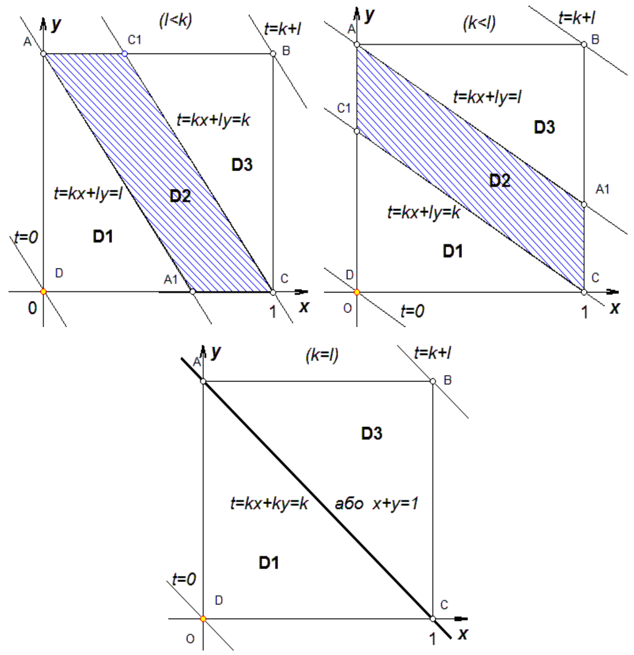

Fig. 1. Splitting the region D into sub-region D_1, D_2, D_3

Then

$$F_{k,l} = I_1 + I_2 + I_3 .$$

The integral I_1 of the region D_1 for the case when $k > l > 0$, reduces to the form:

$$I_1 = \iint_{D_1} f(x,y) e^{-i2\pi(kx+ly)} dxdy =$$
$$= \iint_{\tilde{D}_1} f\left(\frac{kt-lv}{k^2+l^2}, \frac{lt+kv}{k^2+l^2}\right) e^{-i2\pi t} |J| dtdv =$$
$$= \int_0^l \frac{e^{-i2\pi t} dt}{k^2+l^2} \int_{-\frac{lt}{k}}^{\frac{kt}{l}} f\left(\frac{kt-lv}{k^2+l^2}, \frac{lt+kv}{k^2+l^2}\right) dv.$$

Here the value of the Jacobean is taken into account:

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{k}{k^2+l^2} & \frac{-l}{k^2+l^2} \\ \frac{l}{k^2+l^2} & \frac{k}{k^2+l^2} \end{vmatrix} = \frac{1}{k^2+l^2} .$$

Analogously, integrals I_2, I_3 are determined.

Note that the internal integral integration is carried out along the lines $kx + ly = t$. That is, these integrals can be interpreted as projections along a given line, coming from a computer tomograph for each fixed t. Fourier coefficients found in this way are substituted in the Fourier sum to approximate the function $f(x, y)$.

III. Formulas for Calculating Fourier Coefficients of Functions of Three Variables Using a Tomogram

We give formulas for calculating Fourier coefficients of functions of three variables by means of integrals from the approximate function on two-dimensional regions lying in planes parallel to the given base planes.

We assume that the function $f(x,y,z) \in C^r(D)$, $D = [0,1]^3$ is continuous and periodic with a period of one for each of the three variables.

To explain the proposed method, perform the following steps:

Step 1. In the formula for the Fourier coefficients

$$C_{k,l,m}(f) = \iiint_D f(x,y,z) e^{-i2\pi(kx+ly+mz)} dxdydz, \quad -N \le k,l,m \le N, \tag{1}$$

we make the replacement of variables:

$$\begin{cases} kx + ly + mz = t; \\ lx - ky = u; \\ kmx + lmy - (k^2+l^2) z = v. \end{cases} \tag{2}$$

Obviously, the new coordinate system $Otuv$ is Cartesian.

In this case, the Jacobean transition from the system to the system is such:

$$J(t,u,v,k,l,m) = \frac{1}{(k^2+l^2)(k^2+l^2+m^2)}. \tag{3}$$

Step 2. We formulate this step in the form of a theorem, which provides a transition from coordinates t, u, v to coordinates. x, y, z.

Theorem 1. The system (2) has a single solution, which is determined by the formulas:

$$\begin{cases} x = X(t,u,v,k,l,m) = \\ = \dfrac{tk(k^2+l^2) + ul(k^2+l^2+m^2) + vkm}{(k^2+l^2)(k^2+l^2+m^2)} \\ y = Y(t,u,v,k,l,m) = \\ = \dfrac{tl(k^2+l^2) - uk(k^2+l^2+m^2) + vlm}{(k^2+l^2)(k^2+l^2+m^2)} \\ z = Z(t,u,v,k,l,m) = \\ = \dfrac{mt - v}{k^2+l^2+m^2} \end{cases} \tag{4}$$

Step 3. As a result of this coordinate replacement, the formula can be written as:

$$C_{k,l,m}(f) = \\ = \iiint_{D^1} F(t,u,v,k,l,m) e^{-i2\pi t} J(t,u,v,k,l,m)\, du dv dt, \quad (5)$$

where

$$F(t,u,v,k,l,m) = \\ = f\left(X(t,u,v,k,l,m), Y(t,u,v,k,l,m), Z(t,u,v,k,l,m)\right),$$

$$D^1 = \begin{Bmatrix} (t,u,v): 0 \le X(t,u,v,k,l,m) \le 1, \\ 0 \le Y(t,u,v,k,l,m) \le 1,\ 0 \le Z(t,u,v,k,l,m) \le 1 \end{Bmatrix}. \quad (6)$$

We obtain from the double inequalities (6):

$$v_1(t,k,l,m) \le v \le v_2(t,k,l,m), \\ u_1(t,v,k,l,m) \le u \le u_2(t,v,k,l,m).$$

Note that functions:

$$v_1(t,k,l,m), v_2(t,k,l,m), \\ u_1(t,v,k,l,m), u_2(t,v,k,l,m),$$

will be asked by different analytical expressions, depending on the relationship between the indices k,l,m.

That is:

$$C_{k,l,m} = \\ = \int_0^{k+l+m} \left[\int_{v_1(t,k,l,m)}^{v_2(t,k,l,m)} \left[\int_{u_1(t,v,k,l,m)}^{u_2(t,v,k,l,m)} F(t,u,v,k,l,m)\, du \right] dv \right] \times \quad (7) \\ \times e^{-i2\pi t} J(t,u,v,k,l,m)\, dt,$$

In the case when among the indices k,l,m three or two are equal to zero, we will use formulas of an interflatation on a system of mutually perpendicular planes [8]. The case of equality to zero of one of the indices is reduced to the above case for functions of two variables.

IV. Conclusions

1. Integral:

$$\int_{v_1(t,k,l,m)}^{v_2(t,k,l,m)} \left[\int_{u_1(t,v,k,l,m)}^{u_2(t,v,k,l,m)} F(t,u,v,k,l,m)\, du \right] dv$$

for each fixed value t is an integral, from the trace of functions $f(x,y,z)$ in the domain $D_{uv}(t)$ belonging to the plane $kx + ly + mz = t$.

That is numerically such an internal double integral is an integral of a tomogram.

2. Replacement $e^{-i2\pi t}$ of a piecewise-spline of the best approximation [1] allows for finding coefficients $C_{k,l,m}(f)$ to calculate only through integrals of tomograms placed on a system of parallel planes.

3. Replacing the integral for a variable t in formula (7) by quadrature formula with given accuracy, we can put the problem of reducing the number of arithmetic operations in the algorithm as a whole.

4. Using the proposed method to approximate the Fourier sum function $f(x,y,z)$

$$A_N f(x,y,z) = \sum_{k=-N}^{N} \sum_{l=-N}^{N} \sum_{m=-N}^{N} C_{k,l,m} e^{i2\pi(kx+ly+mz)},$$

can proceed to the task of minimizing the number of arithmetic operations for their calculation in the desired grid of nodes in $[0,1]^3$.

5. Similar to the functions of two variables [9, 10], we can investigate a new method of approximating the discontinuous functions of three variables using finite Fourier sums, when known surfaces of the discontinuity of the first kind of an approximate function.

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