Defining logic structures in functional spaces

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Abstract

Many practical problems of geometric information representation and pattern recognition require specific methods and tools for describing complex functions and geometric objects in a way that allows us to use such descriptions for making effective mathematical transformations and logic inferences. Since in real world situations we rarely encounter ideal objects that can be described with the help of one elementary function it is important to develop methods and models for integrating geometric information. There are some original approaches to describing complicated geometric structures, one of which is a method based on so-called R-functions. These functions allow complicated geometric objects to be described analytically, which gives an opportunity to operate with complex objects with the help of a single real function. The concept of R-functions is quite simple but the range of practical applications of this theory is very broad. In this paper we have suggested a generalization of R-functions, which we call *R_p-functions*. Any R-function can be associated with a Boolean function and it is always possible to construct an R-function corresponding to a given Boolean one. Similarly, any R_p-function can be associated with a finite predicate and it is always possible to construct an R_p-function corresponding to a given predicate. If R-functions allow us to use Boolean logic for describing complex objects, R_p-functions provide a possibility of using predicate logic, which is more rich and general than Boolean one. Like R-functions, R_p-functions allow the use of knowledge on what logic rules have been used for constructing a complex object to build a function which is positive inside the given domain and negative outside it. R_p-functions also seem to be interesting by themselves as they allow discovery of logic properties of quite a large class of real functions.

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1 Introduction

Many practical problems of knowledge representation and pattern recognition require specific methods and tools for describing complex functions and geometric objects in a way allowing us to use such descriptions for effective making mathematical transformations and logic inferences. There are some original approaches to describing complicated geometric structures, one of which is a method based on so-called R-functions [1–3]. These functions allow the description of complicated geometric objects analytically, which gives an opportunity to operate with a complex objects with the help of a single real function. The concept of R-functions is quite simple but the range of practical applications of this theory is very broad. The main characteristic property of R-functions is this. They cannot change their sign if their arguments do not change their sign, i.e. the signs of the arguments strictly determine the sign of the function. This simple property allows in some sense treating such functions like Boolean ones. In this paper we give a brief informal introduction to the theory of R-functions and suggest a generalization of these functions where finite predicates are used to discover logic properties of quite a large class of real functions and describe complicated objects with the help of such functions.

2 Properties of R-functions

As it was mentioned above the signs of the arguments of an R-function determine its sign. Consider the function $y = x_1x_2$. This is an R-function as its sign (plus or minus) is determined by the sign of the arguments x_1 and x_2. If we denote "-" as 0 and "+" as 1 it will be possible to construct the following truth table:

Table 1.

x_1	x_2	y
0	0	1
0	1	0
1	0	0
1	1	1

It is easy to see that we have obtained a Boolean equivalence function

$$Y = F(X_1, X_2) = X_1 \sim X_2.$$

In such a way we can construct a Boolean function for any R-function. Since there are an infinite number of R-functions corresponding to a single Boolean function we can talk about "branches" of functions in the functional space. Since the number of Boolean functions with *n* arguments is finite there are a finite number of corresponding branches. It can be shown [2] that

1. If an R-function $y = f(x_1, x_2, \ldots, x_n)$ belongs to a branch $\mathfrak{R}$ and $\varphi(x_1, x_2, \ldots, x_n) > 0$ is an arbitrary function, then $y = f\varphi$ also belongs to $\mathfrak{R}$.
2. The sum of R-functions belonging to a branch $\mathfrak{R}$ also belongs to this branch.
3. Any linear combination with positive coefficients of R-functions belonging to a branch $\mathfrak{R}$ belongs to the same branch.
4. The product of several R-functions is an R-function.
5. If $y = f(u_1, u_2, \ldots, u_m)$ and $u_k = \varphi_k(x_1, x_2, \ldots, x_n)$ $(k = 1, 2, \ldots, m)$ are R-functions then
   $y = f[\varphi_1(x_1, x_2, \ldots, x_n), \varphi_2(x_1, x_2, \ldots, x_n), \ldots, \varphi_m(x_1, x_2, \ldots, x_n)]$
   is an R-function.
6. If $y = f(u_1, u_2, \ldots, u_m)$ is an R-function and functions $u_i = \varphi_i(x_1, x_2, \ldots, x_n)$ $(i = 1, 2, \ldots, m)$ do not change their sign, then the function
   $y = f[\varphi_1(x_1, x_2, \ldots, x_n), \varphi_2(x_1, x_2, \ldots, x_n), \ldots, \varphi_m(x_1, x_2, \ldots, x_n)]$
   does not change its sign.

We can always build a Boolean function corresponding to any R-function. Conversely it is always possible to build an R-function corresponding to a given Boolean function. Nevertheless it has turned out that it is impossible to build an R-function corresponding to the Boolean conjunction ($\wedge$) using only arithmetic operations "+", "-", "$\bullet$". It can be shown that the following functions correspond to the basic Boolean operations of conjunction, disjunction and negation:

$$x_1 \wedge_\alpha x_2 = \frac{1}{2}(x_1 + x_2 - \sqrt{x_1^2 + x_2^2 - 2\alpha x_1 x_2});$$

$$x_1 \vee_\alpha x_2 = \frac{1}{2}(x_1 + x_2 + \sqrt{x_1^2 + x_2^2 - 2\alpha x_1 x_2});$$

$$\bar{x} = -x,$$

where α is an arbitrary function for which $-1 \leq \alpha \leq 1$. The functions $x_1 \wedge_\alpha x_2$, $x_1 \vee_\alpha x_2$ and $\bar{x}$ are called R-conjunction, R-disjunction and R-negation correspondingly.

If $\alpha = \alpha(x_1, x_2)$ is a symmetrical function and $\alpha(-x_1, -x_2) = \alpha(x_1, x_2)$ the following properties are true:

7. $\bar{\bar{x}} = x$
8. $x_1 \wedge_\alpha x_2 = x_2 \wedge_\alpha x_1$
9. $x_1 \vee_\alpha x_2 = x_2 \vee_\alpha x_1$
10. $\overline{x_1 \wedge_\alpha x_2} = \bar{x}_1 \vee_\alpha \bar{x}_2$

11. $\overline{x_1 \vee_\alpha x_2} = \overline{x_1} \wedge_\alpha \overline{x_2}$

12. $x_1 \wedge_\alpha x_2 + x_1 \vee_\alpha x_2 = x_1 + x_2$

13. $(x_1 \wedge_\alpha x_2)(x_1 \vee_\alpha x_2) = \dfrac{1+\alpha}{2} x_1 x_2$

It turns out that it is possible to describe complicated domains in n-dimensional spaces with the help of a single R-function if we know how such domains have been constructed, i.e. if the logic of their construction is known. For example, consider two circles defined by the following inequalities:

$$C_1 : 4 - x^2 - y^2 \geq 0,$$

$$C_2 : 4 - (x-2)^2 - y^2 \geq 0 .$$

Then their intersection can be described by a single inequality with the help of an R-conjunction:

$$C_1 \cap C_2 : (4 - x^2 - y^2) \wedge_\alpha (4 - (x-2)^2 - y^2) \geq 0 .$$

The boundary of the domain $C_1 \cap C_2$ in this case can be described by the following equation:

$$(4 - x^2 - y^2) \wedge_\alpha (4 - (x-2)^2 - y^2) = 0 .$$

In the general case for 2 dimensions the following statement is true [2]:

If domains (D_1), (D_2), …, (D_m) are defined by the following inequalities correspondingly:

$$f_1(x, y) \geq 0, f_2(x, y) \geq 0, ..., f_m(x, y) \geq 0,$$

and the logic of constructing a domain (D) is defined by the Boolean function $D = F(D_1, D_2, \ldots, D_m)$, then the inequality

$$\varphi(x, y) = \varphi[(f_1(x, y), f_2(x, y), ..., f_m(x, y)] \geq 0,$$

where $\varphi(u_1, u_2, ..., u_n)$ is an R-function corresponding to the Boolean function $D = F(D_1, D_2, \ldots, D_m)$, defines the domain (D). The same results are true for any number of dimensions.

Thus R-functions allow using knowledge about how a complicated domain has been constructed by combining simpler domains to obtain a single arithmetic function describing this domain and its boundary.

3 R_p-functions

In the theory of R-functions Boolean algebra plays an important part as the possibility of establishing correspondence between R-functions and Boolean ones gives an opportunity to deal with arithmetic functions in a way very much similar to operating with Boolean ones. In this paper we consider a more general class of functions, which we call R_p- functions. The point is that in the case of

R-functions the sign of a function is determined only by signs of its arguments whereas a natural generalization would be to consider functions that do not change their sign within some predefined intervals.

Let us suppose each of *n* axes is split into a finite number of intervals, which do not overlap:

$$d_{i1} = (-\infty, a_{i1}), d_{i2} = (a_{i1}, a_{i2}), \ldots d_{im_i} = (a_{im_i - 1}, a_{im_i}), d_{im_i + 1} = (a_{im_i}, \infty)$$

where for the axis $x_i : -\infty < a_{i1} < a_{i2} < \ldots < a_{im_i} < \infty, \ i = \overline{1,n}$

If a function $y = f(x_1, x_2, \ldots, x_n)$ does not change its sign within the above intervals we call it an R_p-function. It is easy to see that the set of R_p-functions turns into the set of R-functions if each axis x_i is split exactly into two parts: positive and negative. The main properties of R-functions (1-4) are obviously true also for R_p-functions. Property 5 can be formulated in the following form:

If $y = f(u_1, u_2, \ldots, u_m)$ is an R-function and $u_k = \varphi_k(x_1, x_2, \ldots, x_n)$ $(k = 1, 2, \ldots, m)$ are R_p-functions then

$$y = f[\varphi_1(x_1, x_2, \ldots, x_n), \varphi_2(x_1, x_2, \ldots, x_n), \ldots, \varphi_m(x_1, x_2, \ldots, x_n)]$$

is an R_p-function.

It can be easily shown that for any R_p-function there exists a finite predicate taking on truth values when the function is not negative and false values when it is negative. Consider a simple example:

$f(x,y) = (x-a)(x-b)(y-c)$, where $a<b$. The x-axis is naturally split into three intervals: $d_{11} = (-\infty, a), d_{12} = (a, b), d_{13} = (b, \infty)$. The y-axis is split into two intervals: $d_{21} = (-\infty, c), d_{22} = (c, \infty)$. It is easy to see that f is an R_p-function as it does not change its sign within the above intervals. It is possible to build the following truth table for the finite predicate corresponding to this function:

Table 2.

x	y	f
d_{11}	d_{21}	0
d_{11}	d_{22}	1
d_{12}	d_{21}	1
d_{12}	d_{22}	0
d_{13}	d_{21}	0
d_{13}	d_{22}	1

Using notation of the finite predicate algebra [4] this predicate can be represented in its perfect disjunctive normal form as follows:

$$P(X,Y) = X^{d_{11}} Y^{d_{22}} \vee X^{d_{12}} Y^{d_{21}} \vee X^{d_{13}} Y^{d_{22}}$$

Let us consider now the following problem. How to construct an R_p-function corresponding to a given finite predicate? The answer to this question is quite simple. All we need is just to build an R_p-function corresponding to an arbitrary elementary predicate $X_i^{d_{ij}}$. This function should take on positive values if $x_i \in d_{ij}$ and negative values if it does not. If $j = 1$ we can set $f(x_i) = a_{i1} - x_i$. Obviously $f(x_i)$ > 0 if and only if $x_i \in d_{i1}$. Suppose $1 < j \le m_i$. Then we can set $f(x_i) = (a_{ij} - x_i)(x_i - a_{i,j-1})$. If $j = m_i + 1$ we can set $f(x_i) = (x_i - a_{i,m_i})$.

Any predicate can be obtained with the help of the above elementary comparison operations and the Boolean operations conjunction and disjunction. Since we can now build an R_p-function corresponding to an elementary comparison operation we can use R-conjunction and R-disjunction to construct R_p-functions corresponding to an arbitrary finite predicate. Consider an example. Let us build an R_p-function corresponding to the following predicate:

$$P(X,Y) = X^{d_{12}} \vee Y^{d_{21}}$$

The function $f(x) = (a_{12} - x)(x - a_{11})$ corresponds to the elementary predicate $X^{d_{12}}$ and the function $g(y) = a_{21} - y$ corresponds to the elementary predicate $Y^{d_{21}}$. Therefore the R_p-function

$$s(x,y) = (a_{12} - x)(x - a_{11}) \vee_\alpha (a_{21} - y)$$

corresponds to the predicate $P(X,Y)$.

It turns out that the process of building an R_p-function corresponding to a given predicate can be simplified if we have logic expressions like

$$P(X) = X^{d_{ij1}} \vee X^{d_{ij2}} \vee ... \vee X^{d_{ijk}}$$

If R_p-functions $f_1(x), f_2(x),..., f_k(x)$ correspond to the predicates $X^{d_{ij1}}, X^{d_{ij2}},..., X^{d_{ijk}}$ respectively, the function

$$f(x) = (-1)^{k-1} f_1(x) f_2(x)... f_k(x)$$

corresponds to the predicate $P(X)$. It follows from the fact that if any of the functions $f_1(x), f_2(x),..., f_k(x)$ is positive, $f(x)$ is positive as well since the rest of *k-1* functions are negative. In case all *k* functions $f_1(x), f_2(x),..., f_k(x)$ are negative the sign of $f(x)$ is minus since $(-1)^{2k-1} = -1$.

Finally we would like to mention some cases where R_p-functions can be effectively applied to describing complex geometric objects. Suppose a complex domain is constructed with the help of intersection, union and negation operations and domains described by inequalities like $f_i(x,y) \in d_{ij}$, where intervals d_{ij} do not overlap. Then we can construct an R_p-function corresponding to the finite predicate reflecting the logic of building this domain. This function will be positive inside the domain and negative outside.

4 Conclusions and discussion

Since in real world situations we rarely encounter ideal objects that can be described with the help of one elementary function it is important to develop methods and models for integrating geometric information. R-functions represent a logic viewpoint on complex objects obtained as a result of combining simpler domains. Also these functions are interesting by themselves as they form a class of functions members of which in some aspects behave like Boolean functions. This allows us to better understand the logic structures underlying some real functions. In this paper we have suggested a generalization of R-functions, which we call *R_p-functions*. If R-functions allow us to use Boolean logic for describing complex objects, R_p-functions provide a possibility of using predicate logic, which is more rich and general than Boolean one. We trust that further research into R_p-functions will give us an opportunity to use predicate calculus for generating analytical rules describing complex objects.

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