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Electromagnetic Wave Scattering in a Waveguide Containing Homogeneous Magnetodielectric Spheres*

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A solution is suggested for the problem of electromagnetic wave scattering by an arbitrary number of small resonant-size spheres in a rectangular waveguide. Expressions have been derived for the scattered fields.

The paper is aimed at solving the problem of electromagnetic wave scattering by an arbitrary number of small resonance-size magnetodielectric spheres allocated in an arbitrary order within a rectangular waveguide. The process of scattering by solitary irregularities was analyzed in papers [1,2].

Let N magnetodielectric spheres of radius a_p be located in a rectangular waveguide. The spheres are characterized by permittivities ε_p and permeabilities $\mu_p\left(p \in N\right)$. The waveguide walls lie within the planes $x = 0, x = d; y = 0, y = h$. The Z coordinate is oriented along the waveguide axis. Out of the spheres the inequality holds $a/\lambda << 1$, while inside the resonance case is possible, $a/\lambda_g \sim 1$, where λ and λ_g are wavelengths in free space and the material of a sphere, respectively.

The fields will be represented in the forms $\vec{E}(\vec{r})e^{i\omega t}$ and $\vec{H}(\vec{r})e^{i\omega t}$.

The scattering field can be determined from the knowledge of the internal field in the scattering through the electric, $\vec{\Pi}^E$, and magnetic, $\vec{\Pi}^M$, Hertzian potentials as

$$\begin{aligned}\vec{E}_{sc} &= (\nabla\nabla + \kappa^2\varepsilon_0\mu_0)\vec{\Pi}^E - i\kappa\mu_0\left[\nabla, \vec{\Pi}^M\right],\\ \vec{H}_{sc} &= (\nabla\nabla + \kappa^2\varepsilon_0\mu_0)\vec{\Pi}^M + i\kappa\varepsilon_0\left[\nabla, \vec{\Pi}^E\right].\end{aligned} \tag{1}$$

The Hertz potentials characterizing the dipole portion of the field scattered by a sphere can be represented like

$$\begin{aligned}\vec{\Pi}_p^E &= \hat{f}_p^E \vec{d}_p^E;\\ \vec{\Pi}_p^M &= \hat{f}_p^M \vec{d}_p^M,\end{aligned} \tag{2}$$

where $\hat{f}_p^E, \hat{f}_p^M$ are tensorial Green's functions of the Helmholtz equation for the rectangular waveguide [2,3], and $\vec{d}_p^E, \vec{d}_p^M$ are the dipole moments induced in the spheres by the incident wave, viz.

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$$\vec{d}_p^E = \frac{1}{4\pi}\int_{V_p}\left(\frac{\varepsilon_p}{\varepsilon_0}-1\right)\vec{E}(\vec{r}')d\vec{r}' ;$$

$$\vec{d}_p^M = \frac{1}{4\pi}\int_{V_p}\left(\frac{\mu_p}{\mu_0}-1\right)\vec{H}(\vec{r}')d\vec{r}' .$$

Here $\vec{E}(\vec{r}'),\vec{H}(\vec{r}')$ are the internal fields of the scatterers; V_p is the volume of scatterers P; and ε_0,μ_0 are the permittivity and permeability, respectively, of the waveguide filling material.

The Hertz potentials given by Eq. (2) must satisfy the equations

$$\Delta\vec{\Pi}_p^E+\kappa^2\varepsilon_0\mu_0\,\vec{\Pi}_p^E=-4\pi\delta(|\vec{r}-\vec{r}_0|)\vec{d}_p^E ,$$

$$\Delta\vec{\Pi}_p^M+\kappa^2\varepsilon_0\mu_0\,\vec{\Pi}_p^M=-4\pi\delta(|\vec{r}-\vec{r}_0|)\vec{d}_p^M .$$

First, let us find the internal field of a scatterer for the case where the inequalities $a/\lambda_g <<1$ and $a/\lambda <<1$ hold in side and outside the sphere, respectively. Then the results will be extended to the resonance case where $a/\lambda_g \sim 1$ inside the spheres.

In the space outside the sphere the free-space Green function integrated over the sphere volume gives

$$W(\vec{r})=\int_V \frac{e^{-i\kappa_1|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}dV=\frac{4\pi}{\kappa_1^3}(\sin\kappa_1 a-\kappa_1 a\cos\kappa_1 a)\frac{e^{-i\kappa_1 r}}{r},$$

where $\kappa_1=\kappa\sqrt{\varepsilon_0\mu_0}$; $\kappa=2\pi/\lambda$; and r is the distance from the sphere center to its exterior points. Then, making use of the integrodifferental equations of paper [3] and applying the method of image sources it is possible to construct quasistationary equations for determining the internal fields of the spheres. For a selected sphere p' the equations take the form

$$\vec{E}_{0(p',s'=0,t'=0)}(\vec{r}')=(1-\nabla\nabla\frac{1}{4\pi}\left(\frac{\varepsilon_{p'}}{\varepsilon_0}-1\right)W_{0(p',s'=0,t'=0)})\vec{E}^0_{(p',s'=0,t'=0)}(\vec{r}')-$$
$$-\sum_{\substack{1\\p}}^{N}\sum_{\substack{-\infty\\s}}^{\infty}\sum_{\substack{-\infty\\t}}^{\infty}\Bigg\{\left(\nabla\nabla+\kappa^2\varepsilon_0\mu_0\right)\frac{1}{4\pi}\left(\frac{\varepsilon_p}{\varepsilon_0}-1\right)W^E_{(p,s,t)}(\vec{r})\vec{E}^0_{(p,s=0,t=0)}(\vec{r}')-$$
$$(p,s,t)\neq(p',s'=0,t'=0)$$
$$-i\kappa\mu_0\left[\nabla,\frac{1}{4\pi}\left(\frac{\mu_p}{\mu_0}-1\right)W^M_{(p,s,t)}(\vec{r})\vec{H}^0_{(p,s=0,t=0)}(\vec{r}')\right]\Bigg\},$$

$$\vec{H}_{0(p',s'=0,t'=0)}(\vec{r}')=(1-\nabla\nabla\frac{1}{4\pi}\left(\frac{\mu_{p'}}{\mu_0}-1\right)W_{0(p',s'=0,t'=0)})\vec{H}^0_{(p',s'=0,t'=0)}(\vec{r}')-$$

(3)

$$-\sum_{\substack{1\\p}}^{N}\sum_{\substack{-\infty\\s}}^{\infty}\sum_{\substack{-\infty\\t}}^{\infty}\left\{\left(\nabla\nabla+k^2\varepsilon_0\mu_0\right)\frac{1}{4\pi}\left(\frac{\mu_p}{\mu_0}-1\right)W^M_{(p,s,t)}(\vec{r})\vec{H}^0_{(p,s=0,t=0)}(\vec{r}')+\right.$$

$(p,s,t)\neq(p',s'=0,t'=0)$

$$\left.+i\kappa\varepsilon_0\left[\nabla,\frac{1}{4\pi}\left(\frac{\varepsilon_p}{\varepsilon_0}-1\right)W^E_{(p,s,t)}(\vec{r})\vec{E}^0_{(p,s=0,t=0)}(\vec{r}')\right]\right\},$$

here $\vec{E}_{0(p',s'=0,t'=0)}(\vec{r}')$ and $\vec{H}_{0(p',s'=0,t'=0)}(\vec{r}')$ are the incident wave fields; $\vec{E}^0_{(p',s'=0,t'=0)}(\vec{r}')$ and $\vec{H}^0_{(p',s'=0,t'=0)}(\vec{r}')$ are internal fields of the scatterer; and the values $W_{0(p',s'=0,t'=0)}$, $W^E_{(p,s,t)}(\vec{r})$, $W^M_{(p,s,t)}(\vec{r})$ are

$$W_{0(p',s'=0,t'=0)}=c-\frac{2}{3}\pi r'^2;r'^2=x'^2+y'^2+z'^2,$$

$$W^E_{(p,s,t)}(\vec{r})=\frac{4\pi}{\kappa_1^3}\left(\sin\kappa_1a_p-\kappa_1a_p\cos\kappa_1a_p\right)\frac{e^{-i\kappa_1r_{cc'}}}{r_{cc'}},$$

$$W^M_{(p,s,t)}(\vec{r})=-\frac{4\pi}{\kappa_1^3}\left(\sin\kappa_1a_p-\kappa_1a_p\cos\kappa_1a_p\right)\frac{e^{-i\kappa_1r_{cc'}}}{r_{cc'}}, \tag{4}$$

$$r_{cc'}=\sqrt{\left(x_{p,s}-x_{p',s'=0}\right)^2+\left(y_{p,t}-y_{p',t'=0}\right)^2+\left(z_p-z_{p'}\right)^2},$$

$$x_{p,s}=\left[s-\{(-1)^s-1\}\frac{1}{2}\right]d-(-1)^{s-1}x_{p,s=0},\ (s=0,\pm1,\pm2,...),$$

$$y_{p,t}=\left[t-\{(-1)^t-1\}\frac{1}{2}\right]h-(-1)^{t-1}y_{p,t=0}\ \ (t=0,\pm1,\pm2,...), \tag{5}$$

$$z_p=z_0+l_p=z_0+pl\qquad(p=1,2,3,...,N).$$

The coordinates $x_{p',s'=0},y_{p',s'=0},z_{p'}$ belong to the scatterer p', while $x_{p,s},y_{p,s},z_p$ are coordinates of other spheres and their images in the waveguide walls.

The first terms in the right-hand parts of Eqs. (3) represent the contribution of sphere p' alone, whereas the second term involving triple summation take into account the effect of the rest of the spheres and waveguide walls upon scatterer p'. The effect of waveguide walls is accounted for here in terms of the impacts upon sphere p' of images in the waveguide walls of N spheres. It is assumed that the internal field of a sphere and the field from its image are equal.

With this approach toward accounting for the waveguide wall effect, the problem of estimating the internal field of sphere p' reduces to evaluating the internal field of the sphere p' incorporated in a spatial lattice of N spheres and images of these in the walls of the rectangular waveguide. Each node of the spatial lattice is unambiguously determined by an ordered triad of numbers $c=(p,s,t)$, while its coordinates are given by Eq. (5). In the case of a real sphere the node is at $(p,s=0,t=0)$.

The equations (3) for a solitary sphere, as well as for N spheres, represent a set of $2N$ coupled vectorial equations, or $6N$ scalar equations with $6N$ unknowns in terms of the X-,Y-,Z-components. Its solution for sphere p' takes the form

$$\vec{E}^{0}_{(p',s'=0,t'=0)}(\vec{r}')=\frac{1}{\Delta^{EM}}\left[\sum_{p=1}^{N}\widehat{g}_{p}^{Ep'}\vec{E}_{0(p,s=0,t=0)}(\vec{r}')+\sum_{p=1}^{N}\widehat{\beta}_{p}^{Ep'}\vec{H}_{0(p,s=0,t=0)}(\vec{r}')\right]=\frac{1}{\Delta^{EM}}\left(\Delta_{xc}^{(E)M}\vec{i}+\Delta_{yc}^{(E)M}\vec{j}+\Delta_{zc}^{(E)M}\vec{k}\right),$$

$$\vec{H}^{0}_{(p',s'=0,t'=0)}(\vec{r}')=\frac{1}{\Delta^{EM}}\left[\sum_{p=1}^{N}\widehat{\beta}_{p}^{Mp'}\vec{H}_{0(p,s=0,t=0)}(\vec{r}')+\sum_{p=1}^{N}\widehat{g}_{p}^{Mp'}\vec{E}_{0(p,s=0,t=0)}(\vec{r}')\right]=\frac{1}{\Delta^{EM}}\left(\Delta_{xc}^{(E)M}\vec{i}+\Delta_{yc}^{(E)M}\vec{j}+\Delta_{zc}^{(E)M}\vec{k}\right),\tag{6}$$

where Δ^{EM} is the determinant of the set of $2N$ vectorial equations, Eq. (3); and $\vec{i},\vec{j},\vec{k}$ are unit vectors of the coordinate system.

The Equations (3) for the internal field can be written in a different form. To that end, let us represent the wave incident upon the scattering sphere as an infinite sum of spatial harmonics,

$$\vec{E}_{0(p,s=0,t=0)}(\vec{r}')=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\vec{E}^{mn}_{0(p,s=0,t=0)}(\vec{r}'),$$

$$\vec{H}_{0(p,s=0,t=0)}(\vec{r}')=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\vec{H}^{mn}_{0(p,s=0,t=0)}(\vec{r}')\tag{7}$$

Some of these are propagational waves, while others are exponentially damping away from the sphere. The internal field of the sphere can be written the form of the expansion

$$\vec{E}^{0}_{(p,s=0,t=0)}(\vec{r}')=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\vec{E}^{0mn}_{(p,s=0,t=0)}(\vec{r}'),$$

$$\vec{H}^{0}_{(p,s=0,t=0)}(\vec{r}')=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\vec{H}^{0mn}_{(p,s=0,t=0)}(\vec{r}').\tag{8}$$

Next, let us expand the exponential functions figuring in Eq. (4) in a set of fundamental transverse wave functions of the unloaded waveguide [3], viz.

$$\frac{e^{-i\kappa_1 r_{cc'}}}{r_{cc'}}=\frac{2\pi}{dh}\sum_{mn}\frac{\chi_{mn}}{\beta_{mn}}e^{-i\left[\left(\frac{m\pi}{d}\right)(x_{p,s}-x_{p',s'=0})+\left(\frac{n\pi}{h}\right)(y_{p,t}-y_{p',t'})+\beta_{mn}|z-z_{p'}|\right]}.\tag{9}$$

Then the expressions Eq. (4) for $W^{E}_{(p,s,t)}(\vec{r}),W^{M}_{(p,s,t)}(\vec{r})$ can be brought to the form

$$W^{E}_{(p,s,t)}(\vec{r}) = \frac{8\pi^2}{dh\kappa_1^3}\left(\sin\kappa_1 a_p - \kappa_1 a_p \cos\kappa_1 a_p\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\chi_{mn}}{\beta_{mn}}\, e^{-i\left[\left(\frac{m\pi}{d}\right)(x_{p,s}-x_{p',s'=0})+\left(\frac{n\pi}{h}\right)(y_{p,t}-y_{p',t'=0})+\beta_{mn}\left|z_p-z_{p'}\right|\right]},$$

$$W^{M}_{(p,s,t)}(\vec{r}) = -\frac{8\pi^2}{dh\kappa_1^3}\left(\sin\kappa_1 a_p - \kappa_1 a_p \cos\kappa_1 a_p\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\chi_{mn}}{\beta_{mn}}\, e^{-i\left[\left(\frac{m\pi}{d}\right)(x_{p,s}-x_{p',s'=0})+\left(\frac{n\pi}{h}\right)(y_{p,t}-y_{p',t'})+\beta_{mn}\left|z_p-z_{p'}\right|\right]},$$

where

$$\chi_{mn} = \begin{cases} 2, \text{ with } \; m=0 \text{ or } n=0, \\ 1, \text{ with } \; m,n>0 \end{cases}$$

$$\beta_{mn} = \sqrt{\kappa^2\varepsilon_0\mu_0 - \left(\frac{m\pi}{d}\right)^2 - \left(\frac{n\pi}{h}\right)^2} \text{ with } \quad (m,n=0,1,2,...) \quad (10)$$

The representations Eq. (8) cannot be regarded as the Fourier expansions.

Then Eq. (8) for the $\vec{E}^{0mn}_{(p,s=0,t=0)}(\vec{r}')$ and $\vec{H}^{0mn}_{(p,s=0,t=0)}(\vec{r}')$ components take the form, in the case of an arbitrary sphere,

$$\vec{E}^{mn}_{0(p',s'=0,t'=0)}(\vec{r}') = \left(1 - \nabla\nabla\frac{1}{4\pi}\left(\frac{\varepsilon_{p'}}{\varepsilon_0}-1\right)W_{0(p',s=0,t=0)}\right)\vec{E}^{0mn}_{(p',s'=0,t'=0)}(\vec{r}') -$$

$$-\sum_{\substack{p=1\\(p,s,t)\neq(p',s'=0,t'=0)}}^{N}\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}\frac{2\pi}{d\;h\;\kappa_1^3}(\sin\kappa_1 a_p - \kappa_1 a_p\cos\kappa_1 a_p)\frac{\chi_{\mathrm{mn}}}{\beta_{\mathrm{mn}}}\left\{\left(\nabla\nabla + \kappa^2\varepsilon_0\mu_0\right)\left(\frac{\varepsilon_p}{\varepsilon_0}-1\right)E^{0mn}_{(p,s=0,t=0)}(\vec{r}')\times\right.$$

$$\times e^{-i\left[\left(\frac{m\pi}{d}\right)(x_{p,s}-x_{p',s'=0})+\left(\frac{n\pi}{h}\right)(y_{p,t}-y_{p',t'=0})+\beta_{mn}\left|z_p-z_{p'}\right|\right]} - i\kappa\mu_0\left[\nabla,(-1)\left(\frac{\mu_p}{\mu_0}-1\right)\vec{H}^{0mn}_{(p,s=0,t=0)}(\vec{r}')\times\right.$$

$$\left.\left.\times e^{-i\left[\left(\frac{m\pi}{d}\right)(x_{p,s}-x_{p',s'=0})+\left(\frac{n\pi}{h}\right)(y_{p,t}-y_{p',t'=0})+\beta_{mn}\left|z_p-z_{p'}\right|\right]}\right]\right\},$$

(11)

$$\vec{H}^{mn}_{0(p',s=0,t=0)}(\vec{r}') = \left(1 - \nabla\nabla\frac{1}{4\pi}\left(\frac{\mu_{p'}}{\mu_0}-1\right)W_{0(p',s'=0,t'=0)}\right)\vec{H}^{0mn}_{(p',s'=0,t'=0)}(\vec{r}') -$$

$$
-\sum_{\substack{1\\p}}^{N}\sum_{\substack{-\infty\\s}}^{\infty}\sum_{\substack{-\infty\\t}}^{\infty}\frac{2\pi}{d\ \ h\ \ \kappa_1^3}(\sin\kappa_1 a_p-\kappa_1 a_p\cos\kappa_1 a_p)\frac{\chi_{\mathrm{mn}}}{\beta_{\mathrm{mn}}}\left\{\left(\nabla\nabla+\kappa^2\varepsilon_0\mu_0\right)(-1)\left(\frac{\mu_p}{\mu_0}-1\right)\times\right.
$$

$(p,s,t)\neq(p',s'=0,t'=0)$

$$
\times\vec{H}^{0mn}_{(p,s=0,t=0)}(\vec{r}')e^{-i\left[\left(\frac{m\pi}{d}\right)\left(x_{p,s}-x_{p',s'=0}\right)+\left(\frac{n\pi}{h}\right)\left(y_{p,t}-y_{p',t'=0}\right)+\beta_{mn}\left|z_p-z_{p'}\right|\right]}+
$$

$$
\left.+i\kappa\varepsilon_0\left[\nabla,\left(\frac{\varepsilon_p}{\varepsilon_0}-1\right)\vec{E}^{0mn}_{(p,s=0,t=0)}(\vec{r}')e^{-i\left[\left(\frac{m\pi}{d}\right)\left(x_{p,s}-x_{p',s'=0}\right)+\left(\frac{n\pi}{h}\right)\left(y_{p,t}-y_{p',t'=0}\right)+\beta_{mn}\left|z_p-z_{p'}\right|\right]}\right]\right\}.
$$

The solution of the equation set Eq. (11) for N spheres is

$$
\vec{E}^{0mn}_{(p',s'=0,t'=0)}(\vec{r}')=\frac{1}{\Delta^{mn}}\left[\sum_{p=1}^{N}\widehat{g}_p^{Emnp'}\vec{E}^{mn}_{0(p,s=0,t=0)}(\vec{r}')+\sum_{p=1}^{N}\widehat{\beta}_p^{Emnp'}\vec{H}^{mn}_{0(p,s=0,t=0)}(\vec{r}')\right],
$$

$$
\vec{H}^{0mn}_{(p',s'=0,t'=0)}(\vec{r}')=\frac{1}{\Delta^{mn}}\left[\sum_{p=1}^{N}\widehat{\beta}_p^{Mmnp'}\vec{H}^{mn}_{0(p,s=0,t=0)}(\vec{r}')+\sum_{p=1}^{N}\widehat{g}_p^{Mmnp'}\vec{E}^{mn}_{0(p,s=0,t=0)}(\vec{r}')\right],
\tag{12}
$$

where Δ^{mn} is the determinant of the set of $2N$ vectorial equations Eq. (11).

The number m,n associated with the propagational waves are determined from the conditions

$$
\kappa^2\varepsilon_0\mu_0>\left(\frac{m\pi}{d}\right)^2+\left(\frac{n\pi}{h}\right)^2,\ \text{with}\ (m\le m_0,\ n\le n_0);
\tag{13}
$$

and those corresponding to damping waves from

$$
\kappa^2\varepsilon_0\mu_0<\left(\frac{m\pi}{d}\right)^2+\left(\frac{n\pi}{h}\right)^2,\ \text{with}\ (m>m_0,\ n>n_0).
$$

Then, the internal field of a given spherical scatterer can be represented in the case of scattering by N scatterers, as a sum of terms corresponding to the propagational and damping components, viz.

$$
\begin{aligned}
\vec{E}^{0}_{(p',s'=0,t'=0)}(\vec{r}')&=\sum_{\substack{0\\m}}^{m_0}\sum_{\substack{0\\n}}^{n_0}\frac{1}{\Delta^{mn}}\left[\left(\sum_{p=1}^{N}\widehat{g}_p^{Emnp'}\vec{E}^{mn}_{0(p,s=0,t=0)}(\vec{r}')\right)+\left(\sum_{p=1}^{N}\widehat{\beta}_p^{Emnp'}\vec{H}^{mn}_{0(p,s=0,t=0)}(\vec{r}')\right)\right]+\\
&+\sum_{m>m_0}^{\infty}\sum_{n>n_0}^{\infty}\frac{1}{\Delta^{mn}}\left[\left(\sum_{p=1}^{N}\widehat{g}_p^{Emnp'}\vec{E}^{mn}_{0(p,s=0,t=0)}(\vec{r}')\right)+\left(\sum_{p=1}^{N}\widehat{\beta}_p^{Emnp'}\vec{H}^{mn}_{0(p,s=0,t=0)}(\vec{r}')\right)\right]=\\
&=\vec{E}^{0'}_{(p',s'=0,t'=0)}(\vec{r}')+\vec{E}^{0''}_{(p',s'=0,t'=0)}(\vec{r}');
\end{aligned}
\tag{14}
$$

$$\vec{H}^{0}_{(p',s'=0,t'=0)}(\vec{r}\,') = \sum_{\substack{0\\m}}^{m_0}\sum_{\substack{0\\n}}^{n_0}\frac{1}{\Delta^{mn}}\left[\left(\sum_{p=1}^{N}\hat{\beta}_p^M \vec{H}^{mn}_{0(p,s=0,t=0)}(\vec{r}\,')\right)+\left(\sum_{p=1}^{N}\hat{g}_p^M \vec{E}^{mn}_{0(p,s=0,t=0)}(\vec{r}\,')\right)\right]+$$

$$+\sum_{m>m_0}^{m_0}\sum_{n>n_0}^{n_0}\frac{1}{\Delta^{mn}}\left[\left(\sum_{p=1}^{N}\hat{\beta}_p^M \vec{H}^{mn}_{0(p,s=0,t=0)}(\vec{r}\,')\right)+\left(\sum_{p=1}^{N}\hat{g}_p^M \vec{E}^{mn}_{0(p,s=0,t=0)}(\vec{r}\,')\right)\right]=$$

$$=\vec{H}^{0'}_{(p',s'=0,t'=0)}(\vec{r}\,')+\vec{H}^{0''}_{(p',s'=0,t'=0)}(\vec{r}\,').$$

The solutions obtained in the form of Eqs. (6) and (14) are valid when the inequalities $a/\lambda << 1$ and $a/\lambda_g << 1$ hold outside and inside the sphere, respectively. Meanwhile, they can be extended to thee case where a/λ is small, $a/\lambda << 1$, outside the sphere, while inside $и = \kappa a_p\sqrt{\varepsilon_p\mu_p}$ may assume arbitrary values, including resonance ones. This can be done by means of introduction of effective permeabilities [4] instead of the true values ε_p and μ_p of the sphere. The effective magnitudes, are

$$\begin{aligned}\varepsilon_{peff} &= \varepsilon_p F(\theta),\\ \mu_{peff} &= \mu_p F(\theta),\end{aligned} \qquad (15)$$

with

$$F(\theta) = \frac{2(\sin\theta - \theta\cos\theta)}{(\theta^2-1)\,\sin\theta + \theta\cos\theta}.$$

Fig.1 shows the rums of $\operatorname{Re}F(\theta)$ (solid curve) and $\operatorname{Im}F(\theta)$ (dotted curve) as functions of $\operatorname{Re}\theta$ for several values of the loss tangent, $\tan\delta_\varepsilon$ and μ=1.

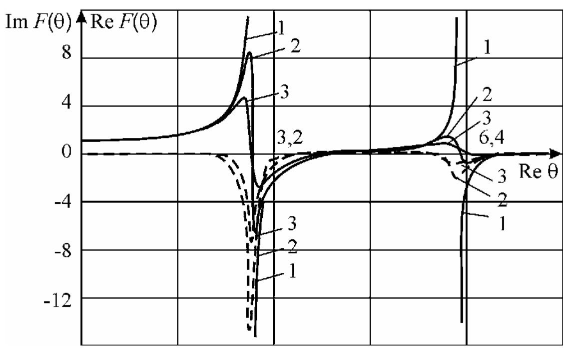

Fig.1. The function $F(\theta)$ for several values of the loss tangent: $\tan\delta_\varepsilon = 0$ (curve 1), $\tan\delta_\varepsilon = 0.05$ (curve 2) and $\tan\delta_\varepsilon = 0.1$ (curve 3).

The dipole moments induced in the spherical scatterer in the case of Eq. (6) can be represented in the form of Eq. (15),

$$\vec{d}_c^E = \frac{1}{4\pi}\left(\frac{\varepsilon_{peff}}{\varepsilon_0} - 1\right)\vec{E}^0_{(p,s=0,t=0)}(\vec{r}')V_p,$$
$$\vec{d}_c^M = \frac{1}{4\pi}\left(\frac{\mu_{peff}}{\mu_0} - 1\right)\vec{H}^0_{(p,s=0,t=0)}(\vec{r}')V_p, \tag{16}$$

while in the case of Eq. (14) they take the form –

$$\vec{d}_p^E = \frac{1}{4\pi}\left(\frac{\varepsilon_{peff}}{\varepsilon_0} - 1\right)\left(\vec{E}^{0'}_{(p,s=0,t=0)}(\vec{r}') + \vec{E}^{0''}_{(p,s=0,t=0)}(\vec{r}')\right)V_p = \vec{d}_p^{E'} + \vec{d}_p^{E''},$$
$$\vec{d}_p^M = \frac{1}{4\pi}\left(\frac{\mu_{peff}}{\mu_0} - 1\right)\left(\vec{H}^{0'}_{(p,s=0,t=0)}(\vec{r}') + \vec{H}^{0''}_{(p,s=0,t=0)}(\vec{r}')\right)V_p = \vec{d}_p^{M'} + \vec{d}_p^{M''}. \tag{17}$$

With account of Eqs. (16) and (17), the Hertz potentials figuring in Eq. (1) can be written in the form of Eq. (2), viz.

$$\vec{\Pi}^E = \sum_{p=1}^{N}\vec{\Pi}_p^E = \sum_{p=1}^{N}\vec{d}_p^E \hat{f}_p^E;$$
$$\vec{\Pi}^M = \sum_{p=1}^{N}\vec{\Pi}_p^M = \sum_{p=1}^{N}\vec{d}_p^M \hat{f}_p^M, \tag{18}$$

where $\hat{f}_p^E, \hat{f}_p^M$ take the form

$$\hat{f}_p^E = \begin{pmatrix} f_{pxx}^E & 0 & 0 \\ 0 & f_{pyy}^E & 0 \\ 0 & 0 & f_{pzz}^E \end{pmatrix}; \ \hat{f}_p^M = \begin{pmatrix} f_{pxx}^M & 0 & 0 \\ 0 & f_{pyy}^M & 0 \\ 0 & 0 & f_{pzz}^M \end{pmatrix}.$$

The components $\hat{f}_p^E, \hat{f}_p^M$ of the tensorial Green functions can be represented in the form of Eq. (10) (reflected wave), viz.

$$f_{pxx}^E = -\frac{2i}{dh}\sum_{m,\,n=0}^{\infty}\frac{\chi_{mn}}{\beta_{mn}}\ \alpha_{p4}\alpha_4 e^{i\beta_{mn}|z-z_p|};$$

$$f_{pyy}^E = -\frac{2i}{dh}\sum_{m,\,n=0}^{\infty}\frac{\chi_{mn}}{\beta_{mn}}\ \alpha_{p3}\alpha_3 e^{i\beta_{mn}|z-z_p|};$$

$$f_{pzz}^E = -\frac{2i}{dh}\sum_{m,\,n=0}^{\infty}\frac{\chi_{mn}}{\beta_{mn}}\ \alpha_{p1}\alpha_1 e^{i\beta_{mn}|z-z_p|};$$

$$f_{pxx}^M = -\frac{2i}{dh}\sum_{m,\,n=0}^{\infty}\frac{\chi_{mn}}{\beta_{mn}}\ \alpha_{p3}\alpha_3 e^{i\beta_{mn}|z-z_p|};$$

$$f_{pyy}^{M}=-\frac{2i}{dh}\sum_{m,\,n=0}^{\infty}\frac{\chi_{mn}}{\beta_{mn}}\;\alpha_{p4}\alpha_{4}e^{i\beta_{mn}|z-z_p|};$$

$$f_{pzz}^{M}=-\frac{2i}{dh}\sum_{m,\,n=0}^{\infty}\frac{\chi_{mn}}{\beta_{mn}}\;\alpha_{p2}\alpha_{2}e^{i\beta_{mn}|z-z_p|},$$

where

$$\alpha_{p1}=\sin\frac{\pi m}{d}x_{p,\,s=0}\sin\frac{\pi n}{h}y_{p,\,t=0},$$

$$\alpha_{p2}=\cos\frac{\pi m}{d}x_{p,\,s=0}\cos\frac{\pi n}{h}y_{p,\,t=0},$$

$$\alpha_{p3}=\sin\frac{\pi m}{d}x_{p,\,s=0}\cos\frac{\pi n}{h}y_{p,\,t=0},$$

$$\alpha_{p4}=\cos\frac{\pi m}{d}x_{p,\,s=0}\sin\frac{\pi n}{h}y_{p,\,t=0},$$

$$\alpha_{1}=\sin\frac{\pi m}{d}x\sin\frac{\pi n}{h}y,$$

$$\alpha_{2}=\cos\frac{\pi m}{d}x\cos\frac{\pi n}{h}y,$$

$$\alpha_{3}=\sin\frac{\pi m}{d}x\cos\frac{\pi n}{h}y,$$

$$\alpha_{4}=\cos\frac{\pi m}{d}x\sin\frac{\pi n}{h}y,$$

Making use of Eqs. (10), (13), (15), and (18), we can obtain from Eq. (1) the expression as follows

$$\vec{E}_{sc}(\vec{r},t)=\sum_{\substack{1\\p}}^{N}\sum_{\substack{0\\m}}^{\infty}\sum_{\substack{0\\n}}^{\infty}\left(-\frac{2i\chi_{mn}}{dh\beta_{mn}}\right)\frac{1}{4\pi}\left[\left(\frac{\varepsilon_{peff}}{\varepsilon_0}-1\right)\widehat{L}_p^{mn}\vec{E}^{0}_{(p,s=0,t=0)}(\vec{r}')-ik\mu_0\left(\frac{\mu_{peff}}{\mu_0}-1\right)\widehat{P}_p^{mn}\vec{H}^{0}_{(p,s=0,t=0)}(\vec{r}')\right]e^{i\left(\omega t+\beta_{mn}|z-z_p|\right)}.$$

(This is the field scattered by the set of N spheres in the waveguide).
Here $\widehat{\mathrm{L}}_c^{mn}$ and $\widehat{\mathrm{P}}_c^{mn}$ are certain functional matrices. In the case where the incident wave is the fundamental mode $H_{10}(m=1,\,n=0)$ they become

$$\widehat{L}_p^{mn}=\begin{bmatrix}0&0&0\\0&\kappa^2\varepsilon_0\mu_0\alpha'_p&0\\0&0&0\end{bmatrix}\text{ and }\widehat{P}_p^{mn}=\begin{bmatrix}0&0&0\\i\beta_{10}\alpha'&0&\frac{\pi}{d}\alpha''_p\\0&0&0\end{bmatrix}$$

with $\alpha'_p=\sin\frac{\pi}{d}x_{p,s=0}\sin\frac{\pi}{d}x$ and $\alpha''_p=\cos\frac{\pi}{d}x_{p,s=0}\sin\frac{\pi}{d}x$.

Hence, the reflection factor of the fundamental mode H_{10} can be written in the form of Eq. (6)

$$\eta = \frac{E_{y\ sc}(\vec{r},t)}{E_{oy\ inc}(\vec{r},t)} = \frac{1}{\left(-ik\frac{d}{\pi}\mu_0 H_0\right)} \left[\frac{1}{\Delta^{EM}} \sum_{p=1}^{N} \left(-\frac{i}{dh\beta_{10}\pi} \right) \left\{ \left(\frac{\varepsilon_{peff}}{\varepsilon_0} - 1 \right) \Delta_{yc}^{(E)M} k^2 \varepsilon_0 \mu_0 \times \right.\right.$$

$$\left. \times \sin\frac{\pi}{d} x_{p,s=0} + k\mu_0\beta_{10} \left(\frac{\mu_{peff}}{\mu_0} - 1 \right) \Delta_{xc}^{E(M)} \sin\frac{\pi}{d} x_{p,s=0} - ik\mu_0 \frac{\pi}{d} \left(\frac{\mu_p}{\mu_0} - 1 \right) \Delta_{zc}^{E(M)} \cos\frac{\pi}{d} x_{p,s=0} \right\} \times \quad (19)$$

$$\left. \times e^{-i\beta_{10}(2z_0 + l_p)} \right] e^{i2\beta_{10}z},$$

where $E_{oy\ inc}(\vec{r},t) = -ik\frac{d}{\pi}\mu_0 H_0 \sin\frac{\pi}{d} x e^{i\left(\omega t - \beta_{10}|z - z_0|\right)}$ and $\beta_{10}^2 = k^2 \varepsilon_0 \mu_0 - \left(\frac{\pi}{d}\right)^2$.

Shown in Fig.2 are the absolute values, $|\eta|$, and phases, φ, of the reflection factor, η, from a sphere, calculated after Eq. (19) in dependence on the radius a of the sphere for $N = 1$; $x_{p,s=0}/d = 0.5$; $y_{p,t=0}/h = 0.5$; $d = 2.3$ cm; $h = 1$ cm; $\lambda = 3$ cm; $\varepsilon_0 = 1$; $\mu_0 = 1$; $\mu = 1$; $\varepsilon' = 400$; and two values of the loss tangent, $\tan\delta_\varepsilon$, namely 0.003 (solid curve) and 0.3 (dotted curve).

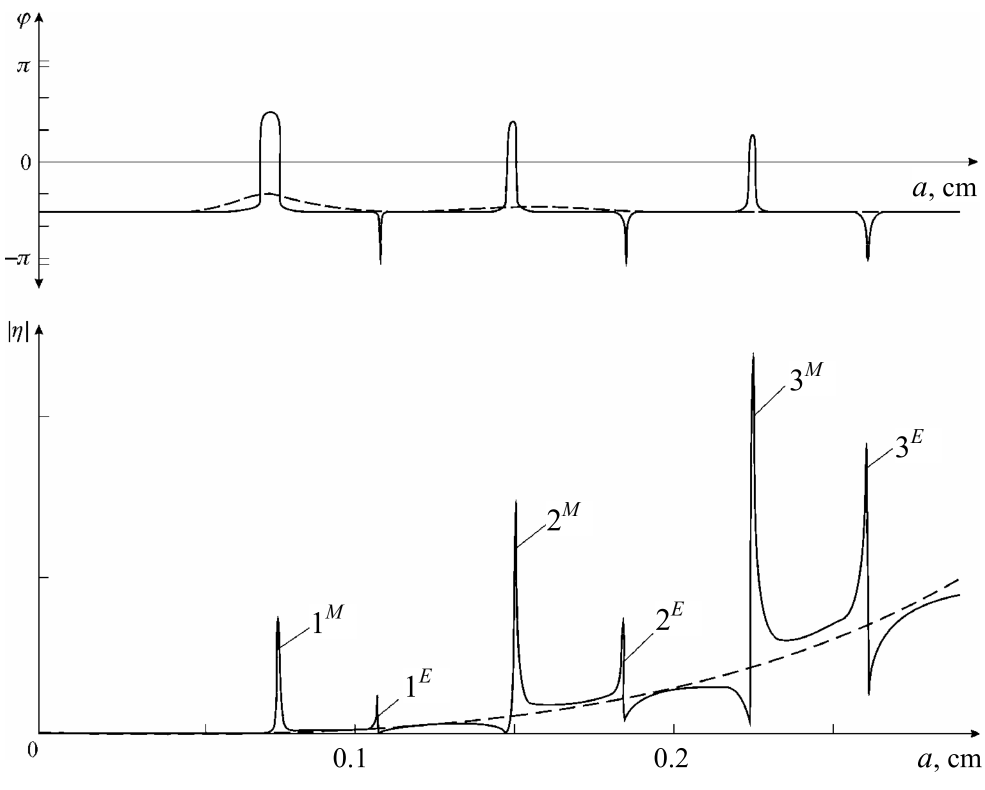

Fig.2. The absolute value, $|\eta|$, and phase, φ, of the reflection factor of the H_{10} mode in dependence on the sphere radius.

The resonances denoted as 1^M, 2^M and 3^m, and $1^E, 2^E$ and 3^E in Fig.2 are, respectively, the magnetic and electric reflection resonances of the first, second and third order. The respective magnetic and electric transmission resonances correspond in Fig.2 to minima of $|\eta|$. The phase jumps of magnitude π in the vicinity of the resonances correspond to the points of minima or maxima of $|\eta|$ where $\tan\delta_\varepsilon = 0$.

As can be seen from Fig.2, the resonance-size sphere is characterized by resonance ranges of different orders, namely, $1^M - 1^E, 2^M - 2^E, 3^M - 3^E, \ldots$, which are zones of order 1, 2, or 3, …, respectively. Each domain involves two resonances of the same order, however of different kind, i.e. magnetic and electric resonances. The domain are bounded from both sides by magnetic and electric transmission resonances and their associated phase jumps.

The resonance structure of the internal field of the sphere in a resonance range is modified as parameters of the sphere or/and external conditions are changed. As a result, resonance effects of either magnetic or electric kind may appear.

As follows from Fig.2, the Q-factor of the magnetic resonances is greater than such of the electric resonances. This can be explained by the difference in the effect of $\tan\delta_\varepsilon$ upon resonances of different kind.

If the sphere in the waveguide are identical and their permittivity, ε_p, and permeability, μ_p, both are real values, then the resonance reflection conditions for the spheres can be found from the equation

$$\det \mathrm{Re}\left\|\alpha_{ij}\right\| = 0, \tag{20}$$

where $\left\|\alpha_{ij}\right\|$ is the principal matrix of the equation set Eq. (3). By resolving Eq. (20) with respect to the function $R(и)$ given by Eq. (17), it is possible to obtain the sought for magnetic and electric resonance conditions for a selected sphere p'. In the case of a free sphere they take the form [1,2] $F(и) = -\frac{2\mu_0}{\mu}$, and $F(и) = -\frac{2\varepsilon_0}{\varepsilon}$, respectively.

A numerical analysis of Eq. (19) in the case of low magnitudes of $\tan\delta_\varepsilon$ shows that the reflection factor curves, η, split into several narrow lines within the range of magnetic and electric resonances. The dynamics of the resonance splitting depends on the location of the spheres in the waveguide [2]. This fine resonance structure of the reflection factor, η, can be used for studying the sphere – waveguide interaction effects.

The pattern of solution suggested in this paper may prove useful for analyzing waveguides loaded by resonance-size spheres.

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