Solving the Wave Equation for a Slotted Cone Placed on an Impedance Plane

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ABSTRACT: Modeling a problem of nonstationary electromagnetic wave diffraction on an infinite circular slotted cone placed on an impedance plane is considered. The solution method is based on using the Laplace inversion, the Kontorovich-Lebedev transforms, and the Riemann-Hilbert method. Representations for the Green's function are obtained.

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INTRODUCTION

Cone structures are canonical ones. There has been interest to problems of electromagnetic wave scattering by cones because of their applications as cone antennas and reflectors in communications and telemetry. There exist works devoted to problems of transient and time-harmonic diffraction by a perfectly conducting isotropic cone [1]. Slots on the screen surface affect the scattered fields. By changing number of slots and their width one can control radiation patterns. Results of studying a time-harmonic scattering wave problem for an alone perfectly conducting cone with longitudinal slots are presented in [2]. In this work the method based on using the Kontorovich-Lebedev integral transforms and an analytical-regularization algorithm for solving the Helmholtz equation for an alone unclosed cone has been developed. The goal of the present paper is to apply this method to solving the wave equation for a slotted cone placed on an impedance plane.

THE PROBLEM STATEMENT AND SOLUTION METHOD (THE THIRD BOUNDARY CONDITION ON A PLANE)

The structure under consideration is a semi-infinite circular perfectly conducting cone Σ_1 with N slots cut along rulings (longitudinal slots) that is placed on the impedance plane Σ_2. In the spherical coordinate system r,ϑ,φ the cone and the plane are defined by equations $\vartheta=\gamma$ and $\vartheta=\pi/2$ respectively. The structure period $l=2\pi/N$ and the slot width of the cone d are angular values. The slot width is a value of the dihedral angle formed by planes passed through the cone axis and cone strip edges.

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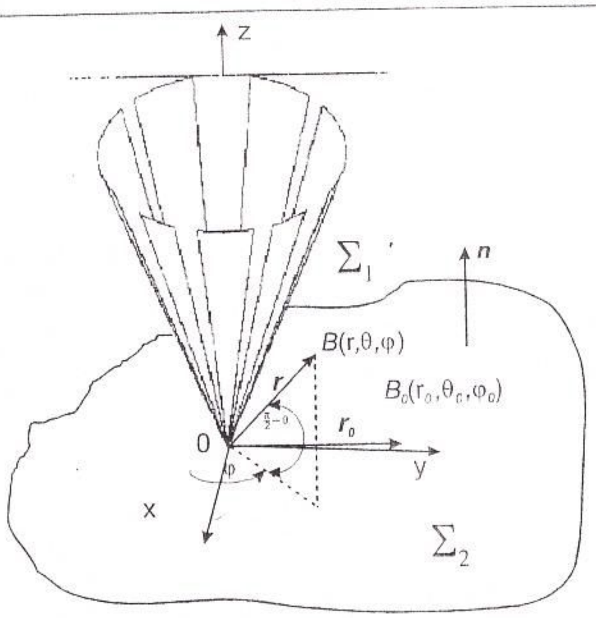

Problem geometry

The time-dependent Green function $G(\vec{r},\vec{r}_0,t,t_0)$ satisfies

i) the partial differential equation

$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\vec{r},\vec{r}_0,t,t_0)=-\delta(\vec{r}-\vec{r}_0)\delta(t-t_0), \tag{1}$$

ii) the boundary conditions

$$G(\vec{r},\vec{r}_0,t,t_0)\big|_{\Sigma_1}=0, \tag{2}$$

$$\left(G+\zeta(\vec{r})\frac{\partial}{\partial n}G\right)\Big|_{\Sigma_2}=0, \tag{3}$$

where $\zeta=\zeta(\vec{r})$ is the given function, $\frac{\partial G}{\partial n}$ is the normal derivative of the function G,

iii) the initial conditions $G=\frac{\partial G}{\partial t}\equiv 0$, at $t<t_0$. (4)

The solution for $G(\vec{r},\vec{r}_0,t,t_0)$ is written as the sum of a free-space field $G_0(\vec{r},\vec{r}_0,t,t_0)$ and a scattered field $G_1(\vec{r},\vec{r}_0,t,t_0)$ due to the presence of the cone and the plane

$$G(\vec{r},\vec{r}_0,t,t_0)=G_0(\vec{r},\vec{r}_0,t,t_0)+G_1(\vec{r},\vec{r}_0,t,t_0), \tag{5}$$

where

$$G_0(\vec{r},\vec{r}_0,t,t_0)=\frac{\delta\left[\tau-|\vec{r}-\vec{r}_0|/c\right]}{4\pi|\vec{r}-\vec{r}_0|},\quad \tau=t-t_0,$$

c is a speed of light in the medium surrounding the cone. We suppose, that $\zeta(\vec{r})=\chi r$, $\chi=\text{const}$.

The time-dependent Green function $G(\vec{r},\vec{r}_0,t,t_0)$ can be obtained via the Laplace inversion from the time-harmonic Green's function $G^s(\vec{r},\vec{r}_0,t_0)$

$$G^s(\vec{r},\vec{r}_0,t_0)=\int_0^{+\infty}G(\vec{r},\vec{r}_0,t,t_0)\exp(-st)dt,\ s>0 \tag{6}$$

that satisfies the three-dimensional wave equation, the boundary conditions like (2) and (3)

$$G^s\big|_{\Sigma_1}=0, \tag{7}$$

$$\left(G^s-\zeta\frac{\partial}{\partial\vartheta}G^s\right)\Big|_{\Sigma_2}=0, \tag{8}$$

the infinity condition at $r\to\infty$, and singularity conditions at the tip and slot edges.

According to (5),

$$G^s(\vec{r},\vec{r}_0,t_0)=G_0{}^s(\vec{r},\vec{r}_0,t_0)+G^s{}_1(\vec{r},\vec{r}_0,t_0), \tag{9}$$

where

$$G_0{}^s(\vec{r},\vec{r}_0,t_0)=\exp(-st_0)\frac{\exp(-q|\vec{r}-\vec{r}_0|)}{4\pi|\vec{r}-\vec{r}_0|},\ q=\frac{s}{c}. \tag{10}$$

The method for solving the stationary boundary problem for G^s uses the Kontorovich–Lebedev integral transforms

$$F(\tau)=\int_0^{+\infty}f(r)\frac{K_{i\tau}(\beta r)}{\sqrt{r}}dr, \tag{11}$$

$$f(r)=\frac{2}{\pi^2}\int_0^{+\infty}\tau\sinh\pi\tau F(\tau)\frac{K_{i\tau}(\beta r)}{\sqrt{r}}d\tau, \tag{12}$$

where $K_\mu(z)$ is the Macdonald function. The function G_0^s can be represented in terms of the Kontorovich–Lebedev transform (11) and (12)

$$G_0^s=\frac{2}{\pi^2}\int_0^{+\infty}\tau\sinh\pi\tau\sum_{m=-\infty}^{+\infty}a_{m\tau}^sU_{m\tau}^0\exp(im\varphi)\frac{K_{i\tau}(qr)}{\sqrt{r}}d\tau, \tag{13}$$

$$a_{m\tau}^s=\frac{(-1)^m}{4}\frac{\exp(-st_0)}{\cosh\pi\tau}\exp(-im\varphi_0)\cdot\frac{\Gamma(1/2-m+i\tau)}{\Gamma(1/2+m+i\tau)}\cdot\frac{K_{i\tau}(qr)}{\sqrt{r}},$$

$$U_{m\tau}^0(\vartheta,\vartheta_0)=\begin{cases}P_{-1/2+i\tau}^m(\cos\vartheta)P_{-1/2}^m(-\cos\vartheta_0),\vartheta<\vartheta_0,\\ P_{-1/2+i\tau}^m(-\cos\vartheta)P_{-1/2}^m(\cos\vartheta_0),\vartheta_0<\vartheta.\end{cases}$$

Taking into account (13), we seek the function G_1^s in the Kontorovich–Lebedev integral form [2]

$$G_1^s = \frac{2}{\pi^2}\int\limits_0^{+\infty} \tau \sinh \pi\tau \sum_{m=-\infty}^{+\infty} b_{m\tau}^s U_{m\tau}^1 \frac{K_{i\tau}(qr)}{\sqrt{r}} d\tau , \quad (14)$$

$$b_{m\tau}^s = -a_{m\tau}^s P_{-1/2+i\tau}^m(\cos\vartheta_0) P_{-1/2+i\tau}^m(-\cos\gamma), \quad 0 < \vartheta_0 < \gamma ,$$

$$U^1{}_{m\tau} = \begin{cases} \sum\limits_{n=-\infty}^{+\infty} \alpha_{mn}(\tau) P_{-1/2+i\tau}^{m+nN}(\cos\vartheta) \exp(i(m+nN)\varphi), 0 < \vartheta < \gamma, \\ \sum\limits_{n=-\infty}^{+\infty} \left[\beta_{mn}(\tau) P_{-1/2+i\tau}^{m+nN}(\cos\vartheta) + \xi_{mn}(\tau) P_{-1/2+i\tau}^{m+nN}(-\cos\vartheta)\right] \exp(i(m+nN)\varphi), \gamma < \vartheta < \pi/2, \end{cases} \quad (15)$$

where α_{mn}, β_{mn}, ξ_{mn} are unknown coefficients, which are independent on parameter q; $P_\mu^m(\cos\vartheta)$ are associated Legendre functions, $\Gamma(z)$ is a gamma-function. To determine unknown coefficients let us use boundary conditions (7), (8) and continuity conditions for G_1^s and its partial derivatives on slots. We arrive at dual series equations (DSE) for x_n connected with unknown coefficients

$$\sum_{n=-\infty}^{+\infty} x_{m,n} \exp(inN\varphi) = \exp(im_0\varphi), \quad \theta < |N\varphi| \le \pi , \quad (16)$$

$$\sum_{n=-\infty}^{+\infty} N(n+\nu)\frac{|n|}{n}(1-\varepsilon_{m,n}) x_{m,n} \exp(in\varphi) = D_{i\tau}^{m_0}, \quad |N\varphi| < \theta , \quad (17)$$

where

$$N(n+\nu)\frac{|n|}{n}(1-\varepsilon_{m,n}) = C_{i\tau}^{(n+\nu)N} \cdot \frac{1}{1-B_{i\tau}^{N(n+\nu)}}, \quad (18)$$

$$C_{i\tau}^M = \frac{(-1)^M}{\pi} \cdot \cosh \pi\tau \cdot \frac{\Gamma(1/2+i\tau+M)}{\Gamma(1/2+i\tau-M)} \cdot \frac{1}{P_{-1/2+i\tau}^M(\cos\gamma) P_{-1/2+i\tau}^M(-\cos\gamma)}, \qquad B_{i\tau}^M = W_{i\tau}^M \cdot \frac{P_{-1/2+i\tau}^M(\cos\gamma)}{P_{-1/2+i\tau}^M(-\cos\gamma)} .$$

$$W_{i\tau}^M = \frac{P_{-1/2+i\tau}^M(0) + \chi \frac{d}{d\vartheta} P_{-1/2+i\tau}^M(\cos\vartheta)\big|_{\vartheta=\pi/2}}{P_{-1/2+i\tau}^M(0) - \chi \frac{d}{d\vartheta} P_{-1/2+i\tau}^M(\cos\vartheta)\big|_{\vartheta=\pi/2}}, \qquad D_{i\tau}^p = N(p+\nu) \cdot \frac{|p|}{p}(1-\varepsilon_{m,p}) \cdot B_{i\tau}^{N(p+\nu)}, \qquad \theta = \pi d/l ,$$

$m/N = \nu + m_0, -1/2 \le \nu < 1/2$, m_0 is the nearest integer to m/N. One can verify that

$$\varepsilon_{m,n} = O\left(\frac{1}{N^2(n+\nu)^2}\right), N(n+\nu) >> 1. \quad (19)$$

By using the Riemann-Hilbert problem method for a unit circle arc, we bring DSE to the system of linear algebraic equations (SLAE) for coefficients x_n. For any problem parameters the solution to SLAE can be obtained with the reduction method and for a semitransparent cone $(N >> 1, (l-d)/l << 1)$ with the iteration one by virtue (19). The inversion of G^s is accomplished by procedure in [1]. It follows that

$$G_1 = \frac{c}{4\pi r r_0}\eta\left(\hat{t} - \frac{r+r_0}{c}\right)\sum_{m=-\infty}^{+\infty}\exp(im\varphi)\int_0^{+\infty} g_{m\tau}\, U^1_{m\tau}\cdot P_{-1/2+i\tau}(\cosh b)d\tau\,, \tag{20}$$

$$g_{m\tau} = (-1)^{m+1}\exp(-im\varphi_0)\tau\tanh\pi\tau\cdot\frac{\Gamma(1/2-m+i\tau)}{\Gamma(1/2+m+i\tau)}P^m_{-1/2+i\tau}(\cos\vartheta_0)P^m_{-1/2+i\tau}(-\cos\gamma)\,,$$

$$\cosh b(\hat{t}) = \frac{\hat{t}^2c^2 - r^2 - r_0^2}{2rr_0}\,,$$

where $\eta(z)$ is the Heaviside unit function.

ANALYTICAL SOLUTIONS FOR A SEMITRANSPARENT CONE

To simplify the general results we assume that the source point is located on the cone axis, i.e., $\vartheta_0 = 0$, $\varphi_0 = 0$. For a semitransparent cone that is defined by existence of the limit

$$\lim_{\substack{N\to+\infty\\ d/l\to 1}}\left[-\frac{1}{N}\ln(1-d/l)\right] = Q > 0\,,$$

we obtain from (20)

$$G_1 = -\frac{c}{4\pi r r_0}\eta\left(\hat{t}-\frac{r+r_0}{c}\right)\int_0^{+\infty}\tau\tanh\pi\tau W_{i\tau}P_{-1/2+i\tau}(\cos\vartheta)\cdot P_{-1/2+i\tau}(\cosh b)d\tau +$$

$$\frac{c}{4\pi r r_0}\eta\left(\hat{t}-\frac{r+r_0}{c}\right)\int_0^{+\infty}\tau\tanh\pi\tau\frac{A_{i\tau}}{2Q+A_{i\tau}}\left[W_{i\tau}P_{-1/2+i\tau}(\cos\vartheta) - P_{-1/2+i\tau}(-\cos\vartheta)\right]\cdot P_{-1/2+i\tau}(\cosh b)d\tau\,,$$

$$\gamma < \vartheta < \pi/2\,, \tag{21}$$

$$A_{i\tau} = \frac{\pi}{ch\pi\tau}\left(P_{-1/2+i\tau}(-\cos\gamma) - P_{-1/2+i\tau}(\cos\gamma)\right)P_{-1/2+i\tau}(\cos\gamma)\,,$$

$$W_{i\tau} = W^M_{i\tau}\big|_{M=0}\,. \tag{22}$$

By using the residue theorem one can derive series representations for G_1. The boundary problem spectrum for the semitransparent cone placed on the impedance plane is defined by roots $\mu_j = \mu_j(\gamma, Q, \chi)$ of the equation $A_\mu + 2Q = 0$. One may determine these roots asymptotically for the special types of the semitransparent cone

1. $Q \ll 1$,

$$\mu_j^+ = \alpha_j^+ - \frac{2Q\cos\pi\alpha_j^+}{\pi P_{-1/2+\alpha_j^+}(-\cos\gamma)\frac{d}{d\mu}P_{-1/2+\mu}(\cos\gamma)\Big|_{\mu=\alpha_j^+}} + O(Q^2),$$

$$P_{-1/2+\alpha_j^+}(\cos\gamma) = 0,$$

$$\tilde{\mu}_j = \varsigma_j - 2Q\frac{\cos\pi\varsigma_j\left[P_{-1/2+\varsigma_j}(0) - \chi\frac{d}{d\vartheta}P_{-1/2+\varsigma_j}(\cos\vartheta)\Big|_{\vartheta=\pi/2}\right]}{\pi P_{-1/2+\varsigma_j}(\cos\gamma)\frac{d}{d\mu}V_\mu\Big|_{\mu=\varsigma_j}} + O(Q^2),$$

$$V_\mu = \left[P_{-1/2+\mu}(-\cos\gamma) - P_{-1/2+\mu}(\cos\gamma)\right]P_{-1/2+\mu}(0) -$$
$$-\chi\left[P_{-1/2+\mu}(-\cos\gamma) + P_{-1/2+\mu}(\cos\gamma)\right]\frac{d}{d\vartheta}P_{-1/2+\mu}(\cos\vartheta)\Big|_{\vartheta=\pi/2},$$

$$V_{\varsigma_j} = 0,\ \varsigma_j = \varsigma_j(\gamma, \chi),$$

2. $Q \gg 1$,

$$\mu_j = \upsilon_j + \frac{1}{2Q}\frac{\pi P_{-1/2+\upsilon_j}(\cos\gamma)}{\cos\pi\upsilon_j\frac{d}{d\mu}\Phi_\mu\Big|_{\mu=\upsilon_j}}\left[P_{-1/2+\upsilon_j}(\cos\gamma)\right]^2\left[P_{-1/2+\upsilon_j}(0) + \chi\left(\upsilon_j^2 - \frac{1}{4}\right)P_{-1/2+\upsilon_j}(0)\right] + O(Q^{-2}),$$

$$\Phi_\mu = P_{-1/2+\mu}(0) - \chi\left(\mu^2 - \frac{1}{4}\right)P^{-1}_{-1/2+\mu}(0),\ \Phi_{\upsilon_y} = 0.$$

The factor $W_{i\tau}$ (22) in the integral representation (21) for G_1 describes the impedance plane effects.

THE FOURTH BOUNDARY CONDITION ON A PLANE

Let us consider the problem for finding the Green function $G(\vec{r}, \vec{r}_0, t, t_0)$ that satisfies the equation (1), the Dirichlet boundary condition (2) on the cone, the fourth boundary condition on the plane

$$\left(\zeta_1\frac{\partial G}{\partial n} + \zeta_2\frac{\partial G}{\partial \varsigma}\right)\Big|_{\Sigma_2} = 0,\ \zeta\cdot\zeta_2 \neq 0, \tag{23}$$

and the initial conditions (4). Functions $\frac{\partial G}{\partial n}$ and $\frac{\partial G}{\partial \varsigma}$ are normal and tangent derivatives respectively.

$$\zeta_1 = \zeta_1(r,\varphi),\ \zeta_2 = \zeta_2(r,\varphi),\ \vec{n} = -\vec{e}_\vartheta,\ \vec{\varsigma} = \alpha_r\vec{e}_r + \beta_\varphi\vec{e}_\varphi. \tag{24}$$

Taking into account (24) one can write (23) in the form

$$\left(\zeta_2 r\alpha_r \frac{\partial G}{\partial r} - \zeta_1 \frac{\partial G}{\partial \vartheta} + \zeta_2 \beta_\varphi \frac{\partial G}{\partial \varphi}\right)\Big|_{\Sigma_2} = 0 . \tag{25}$$

Let $\alpha_r = 0$, then (25) becomes

$$\left(-\zeta_1 \frac{\partial G}{\partial \vartheta} + \zeta_2 \beta_\varphi \frac{\partial G}{\partial \varphi}\right)\Big|_{\Sigma_2} = 0, \quad \zeta_1 = \zeta_1(r,\varphi), \zeta_2 = \zeta_2(r,\varphi). \tag{26}$$

The following cases are possible:

$\zeta_1 = \chi_1 = \text{const}$, $\zeta_2\beta_\varphi = \chi_2 = \text{const}$, $\beta_\varphi(r,\varphi) \neq 0$,

$\zeta_1 = \chi_1 \cdot \psi(r,\varphi)$, $\chi_1 = \text{const}$, $\zeta_2\beta_\varphi = \chi_2 \cdot \psi(r,\varphi)$, $\chi_2 = \text{const}$, $\psi(r,\varphi) \neq 0$,

$\zeta_1 = \chi_1 \cdot \beta_\varphi(r,\varphi)$, $\chi_1 = \text{const}$, $\zeta_2 = \chi_2 = \text{const}$.

Thus, (26) is reduced to

$$-\chi_1 \frac{\partial G}{\partial \vartheta} + \chi_2 \frac{\partial G}{\partial \varphi} = 0, \quad \vartheta = \pi/2, \ r \in (0,+\infty), \varphi \in [-\pi,\pi], \tag{27}$$

provided that χ_1 and χ_2 are constant. Now suppose that the boundary condition (27) is given.

For finding the Green function $G(\vec{r},\vec{r}_0,t,t_0)$ the above mentioned method can be used ((5)–(15)). It leads to DSE for Fourier coefficients $\hat{x}_{m,n}$ of the following form:

$$\sum_{n=-\infty}^{+\infty} \hat{x}_{m,n} \exp(inN\varphi) = \exp(im_0\varphi), \quad \theta < |N\varphi| \le \pi ,$$

$$\sum_{n=-\infty}^{+\infty} N(n+\nu)\frac{|n|}{n}(1-\hat{\varepsilon}_{m,n})\hat{x}_{m,n} \exp(in\varphi) = \hat{D}_{i\tau}^{m_0}, \quad |N\varphi| < \theta ,$$

where

$$N(n+\nu)\frac{|n|}{n}(1-\hat{\varepsilon}_{m,n}) = C_{i\tau}^{(n+\nu)N} \cdot \frac{1}{1-\hat{B}_{i\tau}^{N(n+\nu)}},$$

$$\hat{B}_{i\tau}^{M} = \hat{W}_{i\tau}^{M} \cdot \frac{P_{-1/2+i\tau}^{M}(\cos\gamma)}{P_{-1/2+i\tau}^{M}(-\cos\gamma)},$$

$$\hat{W}_{i\tau}^{M} = \frac{\chi_2 iM \cdot N \cdot P_{-1/2+i\tau}^{M}(0) + \chi_1 \frac{d}{d\vartheta} P_{-1/2+i\tau}^{M}(\cos\vartheta)\big|_{\vartheta=\pi/2}}{\chi_2 iM \cdot N \cdot P_{-1/2+i\tau}^{M}(0) - \chi_1 \frac{d}{d\vartheta} P_{-1/2+i\tau}^{M}(\cos\vartheta)\big|_{\vartheta=\pi/2}},$$

$$\hat{D}_{i\tau}^{p} = N(p+\nu)\cdot\frac{|p|}{p}(1-\hat{\varepsilon}_{m,p})\cdot \hat{B}_{i\tau}^{N(p+\nu)},$$

$$\hat{\varepsilon}_{m,n} = O\left(\frac{1}{N^2(n+\nu)^2}\right), N(n+\nu) >> 1.$$

These DSE (as (16) and (17)) can be reduced to SLE [2]. The solutions of SLE are obtained both numerically and analytically.

CONCLUSION

The method for solving the wave equation for a perfectly conducting infinite circular periodically slotted cone placed on an impedance plane is presented. Two special boundary conditions on the plane are considered. They are the third (mixed) boundary condition and the fourth boundary condition (connection between the normal derivative of the function and the tangent one). This method uses the Laplace inversion, the Kontorovich-Lebedev integral transforms and Riemann-Hilbert problem method. By virtue of it the given problem is reduced to the system of linear algebraic equations that can be solved both analytically (for special cases of the slotted cone) and numerically. Representations for the Green function for the semitransparent cone placed on the impedance plane with the third boundary condition are obtained.

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