# Modeling of Frequency Conversion in a Chain of Coupled Resonators due to Time Change in Permittivity

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*Abstract* — **This paper considers the transformation of natural modes of single and coupled dielectric resonators when their material is subject to an abrupt time change in permittivity. Both the transient response and the new steady state regime are described in detail. Possibility of frequency shift is demonstrated. Enhancement of the frequency shift for the coupled modes with odd-odd symmetry in the chain has been shown.**

*Index Terms* — **Dielectric resonators, time-varying media, whispering gallery modes.**

## I. Introduction

Dielectric resonators with whispering-gallery modes (WGM) have been the subject of significant interest in recent years as they exhibit properties useful for a wide range of applications, including low-threshold lasers [1], ultra-small filters [2] and sensors [3, 4]. However, much of the theoretical work focused upon resonators has concentrated upon prediction of their frequency domain properties, although accurate time domain modeling is essential for microwave design, especially for active devices and circuit components.

The most widely used today numerical approach is FDTD method that is flexible but demands large computer memory. Moreover, conventional FDTD codes have problems with visualization of the high Q resonances.

In this paper we use rigorous mathematical method that allows us to analyze problems both in the frequency domain and in the time domain. Applying the Laplace transform directly to the wave equation we derive an analytical solution of the problem in the frequency domain. Then we recover the time domain electromagnetic field by virtue of the computation of the inverse Laplace transform via the residue evaluation at singular points associated with eigenvalues of the structure. This approach guarantees accurate back transformation with controllable accuracy and allows us to extract and interpret physical phenomena easily. This method has been already successfully applied to a variety of time domain problems with different geometries [5-7].

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In this paper we investigate the transient response of a WGM of the resonator to an abrupt change in permittivity. Proposed approach is extended then to the case of WGM transformation in a finite linear chain of coupled circular dielectric resonators due to temporal changes in their permittivity. In practice, temporal changing of the material constants index can be realized by varying an input signal in a nonlinear structure [8]; by voltage control [9]; by a focused laser beam as a local heat source [10] or else by a plasma injection [11]. Note that the temporal variations of the permittivity of an unbounded medium transform the frequency of an existing monochromatic field, however both the wavenumber and the field pattern are conserved [12, 13].

In this paper, the details of the field evolution during the transient period in a single resonator and in a linear chain of coupled resonators will be characterized.

## II. Mathematical Formulation

### *A. Single resonator*

We consider a circular dielectric resonator of radius $a$. This can be viewed as a 2D model of a thin-disk 3D resonator within the effective refractive-index approximation. Dielectric permittivity of the material is $\varepsilon_1$, surrounding medium is a vacuum. The material is considered to be linear and non-magnetic. To describe the fields, the cylindrical system of coordinates $\rho, \varphi, z$, centered on the resonator, is introduced. Before a change in the permittivity, an initial field exists in the resonator which is an H-polarized natural mode as this type of modes is dominant in thin disks; the z-component of it can be represented in the following form

$$H_z = A\begin{cases} b_k J_k(k_1\rho)\cos k\varphi, & \rho < a \\ H_k^{(2)}(k_0\rho)\cos k\varphi, & \rho > a, \end{cases} \tag{1}$$

wher $b_k = H_k^{(2)}(k_0 a)\big/J_k(k_1 a)$, $k_0 = \omega_0/c$, $k_1 = \sqrt{\varepsilon_1}\,\omega_0/c$, c is light velocity in vacuum, $\omega_0$ is complex valued eigenfrequency that is the solution of the equation

$$J_k(k_1 a)H_k'^{(2)}(k_0 a) - \frac{k_0}{k_1}J_k'(k_1 a)H_k^{(2)}(k_0 a) = 0 \tag{2}$$

The time dependence is assumed as $e^{i\omega_0(t-t^*)}\Theta(t-t^*)$, where $t^*<0$ is the moment of switching on the mode. At zero moment of time the dielectric permittivity inside the resonator changes abruptly in value from $\varepsilon_1$ to $\varepsilon_2$ in response to an external force. We now investigate the mechanisms that couple the initial mode to those of the cavity with the new permittivity, with particular emphasis on the transient processes occurring in such a single dynamic resonator. The formulation of the problem in the above manner permits construction of an analytical solution that explains the interesting phenomena in detail.

The transformed field has to satisfy the wave equations

$$\begin{aligned}&\Delta H-\frac{\varepsilon_2}{c^2}\frac{\partial^2}{\partial t^2}H=0,\ \rho<a,\\&\Delta H-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}H=0,\ \rho>a.\end{aligned}\tag{3}$$

Here H represents $H_z$ component of the field which is perpendicular to the plane of the resonator and $\Delta$ is the Laplace operator

$$\Delta=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}.$$

We apply the Laplace transform

$$L(p)=\int_0^\infty W(t)e^{-pt}dt$$

directly to the wave equation (3).

The electric field flux density, as well as the magnetic induction, remains continuous at time jumps of the medium parameters. It follows that the initial conditions in transient region have the form

$$H(t=0^+)=H(t=0^-),\ \frac{\partial}{\partial t}H(t=0^+)=\frac{\varepsilon_1}{\varepsilon_2}\frac{\partial}{\partial t}H(t=0^-).\tag{4}$$

Here we adopt the evolutionary approach presented in [6]. Within this approach, the complete transient solution is explicitly constructed as a superposition of the waves reflected from the structure boundaries; the field at an instant just after the moment of switching being evaluated as if the transient medium is unbounded. Therefore, because of the finite speed of the electromagnetic waves, the influence of the resonator boundary appears only after a finite time delay from the moment of switching and this will be discussed further.

We will seek the solution in the form

$$\begin{aligned}&L_{in}(p)=Ab_k\cos k\varphi\frac{v_1^2p+i\omega_0v_2^2}{p^2v_1^2+\omega_0^2v_2^2}J_k(k_1\rho)e^{-pt^*}+\\&+\cos k\varphi B_k(p)I_k(\frac{p}{v_2}\rho),\ \rho<a\end{aligned}\tag{5}$$

$$\begin{aligned}&L_{out}(p)=A\cos k\varphi H_k^{(2)}(k_0\rho)\frac{e^{-pt^*}}{p-i\omega_0}+\\&+\cos k\varphi C_k(p)K_k(\frac{p}{c}\rho),\ \rho>a,\end{aligned}\tag{6}$$

where $v_1=c/\sqrt{\varepsilon_1}$, $v_2=c/\sqrt{\varepsilon_2}$. The complete field within the cavity consists of the unbounded term (first term in (5)) and additional contributions due to the boundary. Similarly, in the outer region the field comprises a superposition of the initial field (first term in (6)) and contributions due to the presence of the boundary.

Outside the cavity, the function $K_k(..)$ guarantees satisfaction of the Sommerfeld outgoing radiation condition at infinity, and $B_k$ and $C_k$ are unknown coefficients to be determined, which are chosen so that at the cylindrical boundary, $\rho=a$, the tangential components of the field are continuous, i.e.

$$\begin{aligned}&H_z(\rho=+a)=H_z(\rho=-a),\\&E_\varphi(\rho=+a)=E_\varphi(\rho=-a)\end{aligned}\tag{7}$$

Unknown coefficients can be presented in the form

$$B_k=\frac{\omega\sqrt{\varepsilon_1\varepsilon_2}J_k(k_1a)K_k'(\frac{p}{c}a)+p\sqrt{\varepsilon_2}J_k'(k_1a)K_k(\frac{p}{c}a)}{\sqrt{\varepsilon_1}I_k'(\frac{p}{v_2}a)K_k(\frac{p}{c}a)-\sqrt{\varepsilon_1\varepsilon_2}K_k'(\frac{p}{c}a)I_k(\frac{p}{v_2}a)}U(p),$$

$$C_k=\frac{p\sqrt{\varepsilon_2}J_k'(k_1a)I_k(\frac{p}{v_2}a)+\omega\sqrt{\varepsilon_1}I_k'(\frac{p}{v_2}a)J_k(k_1a)}{\sqrt{\varepsilon_1}I_k'(\frac{p}{v_2}a)K_k(\frac{p}{c}a)-\sqrt{\varepsilon_1\varepsilon_2}K_k'(\frac{p}{c}a)I_k(\frac{p}{v_2}a)}U(p),\tag{8}$$

$$U(p)=\frac{ip(v_2^2-v_1^2)}{(p-i\omega_0)(p^2v_1^2+\omega_0^2v_2^2)}b_ke^{-pt^*}.$$

The expressions have the poles and the branch-cut along the negative real axis of the complex p-plane. There are poles associated with the frequency of the initial wave $p=i\omega_0$ and the transformed frequency $p=\pm i\,v_2/v_1\,\omega_0$ due to the permittivity changing. There is also an infinite number of poles associated with the zeros of the denominator $B_k$ and $C_k$ in (8). They correspond to the eigenfrequencies of the resonator in its new state.

Using the asymptotic expansions for modified Bessel functions with large arguments, we have

$$pB_k I_k\left(\frac{p}{v_2}\rho\right) \sim i\frac{v_2^2 - v_1^2}{v_1^2}\frac{c}{c+v_2} b_k\sqrt{\frac{a}{\rho}} J_k'(k_1 a) e^{\frac{p}{v_2}(\rho - a)} \quad (9)$$

$$pC_k K_k\left(\frac{p}{c}\rho\right) \sim i\frac{v_2^2 - v_1^2}{v_1^2}\frac{c}{c+v_2} b_k\sqrt{\frac{a}{\rho}} J_k'(k_1 a) e^{-\frac{p}{c}(\rho - a)} \quad (10)$$

From (9) and (10) it is observed that, upon inversion to the time domain, the expressions corresponding to $B_k(p)I_k(\rho p/v_2)$ and $C_k(p)K_k(\rho p/c)$ exhibit a time delay that can be expressed in terms of unit-step Heaviside functions, $\Theta(v_2 t + \rho - a)$ and $\Theta(ct - \rho + a)$, inside and outside of the resonator, respectively.

In the "early time" regime ( $t < a/v_2$ ) inside the resonator the field is described by the first term in (5) and exhibits the same wavenumber and shifted frequencies predicted from the abrupt change in the material parameters. It does not depend upon the boundary shape. In the outer region, only the initial field is present. Near the boundary region, the transient waves appear that correspond to total field given in (5) and (6). In the time domain they are expressible in terms of a residue sum over all the singular points and an integral along the branch cut. Examination of (5) and (6) in more detail reveals that the singularities in the total field at $p = \pm i v_2/v_1 \omega_0$ and $p = i\omega_0$ do not contribute to the residue sum. It is also confirmed that there is a term in the transient response inside the resonator that provides immediate cancellation of the primary wave.

In the E polarization case, the problem can be solved in similar way – see [14, 15].

### *B. Linear chain of resonators*

Proposed approach can be extended to a finite linear chain of coupled circular dielectric resonators. We consider the 2D model of the linear chain of N identical circular resonators with the radii a. The separation distance between the resonators is $d$, the dielectric permittivity of the material is $\varepsilon_1$. The transversal electric (H-polarized) WGM is considered as an initial field; the z-component of it can be represented in the following form

$H_z(\rho_j, \varphi_j) = \sum_{s=-\infty}^{+\infty} A_s^{(j)} J_s(k_1 \rho_j) e^{is\varphi_j}$ inside the $j^{th}$ resonator,

$H_z(\rho_j, \varphi_j) = \sum_{j=2}^{N} \sum_{s=-\infty}^{+\infty} \bar{A}_s^{(j)} H_s^{(2)}(k_0 \rho_j) e^{is\varphi_j}$ in outer space.

Here $(\rho_j, \varphi_j, z)$ , $j = 2...N$ is a set of N cylindrical systems of coordinates associated with each resonator, coefficients $A_s^{(j)}$ , $\bar{A}_s^{(j)}$ can be found to satisfy corresponding boundary conditions (7) at each boundary as discussed in [16, 17].

At zero moment of time, the dielectric permittivity is changed abruptly in the whole structure from the value $\varepsilon_1$ to the value $\varepsilon_2$. The field after the medium change satisfies the wave equations (3) written for each particular resonator and surrounding space. We solve this problem similarly to single-resonator case and apply the Laplace transform directly to the wave equations including the initial and boundary conditions at the circular interfaces. Using the addition theorem for the Bessel functions, we arrive at an infinite set of algebraic equations that can be truncated in order to provide a predetermined numerical precision. The resulting field in the time domain is obtained using the inverse Laplace transform.

## III. Numerical Results

Here, we introduce dimensionless values: $w_0 = a\omega_0/c$ is the normalized frequency, $T = tc/a$ is the normalized time, $r = \rho/a$ is the normalized distance.

To estimate duration of transient (also called ring-time) period in a single resonator, we plot the time dependence of the field inside the resonator. Here, $WGH_{8,1}$ mode is considered as initial field with $w_0 = 4.5418 + 0.000399i$, and refractive index is $n_1 = \sqrt{\varepsilon_1} = 2.631$. At zero moment of time, the refractive index changes to the value $n_2 = \sqrt{\varepsilon_2} = 2.63$.

Fig. 1 presents the total field normalized by the maximum of amplitude of the initial field versus the normalized time near the centre of the resonator (r=0.05). Changing the refractive index leads to the excitation of all modes with the same angular dependence as initial one however with growing of *r* transient process becomes smooth (Fig. 2). Evaluating the residues at each singular pole, we conclude that maximum amplitude has the mode with the same field pattern as initial one as seen in Fig. 2. The change of the refractive index leads to the frequency shift of the mode from the initial value $\omega_0 = \omega_0' + i\omega_0''$ to the transformed value $\omega_1 = \omega_1' + i\omega_1''$.

Fig. 3 represents the absolute value of the normalized frequency shift $\Delta w = \Delta\omega a/c$ ( $\Delta\omega = \omega_1' - \omega_0'$ ) of the transformed modes with different numbers of angular variations. It is seen that WGM with a greater Q-factor demonstrates a greater frequency shift.

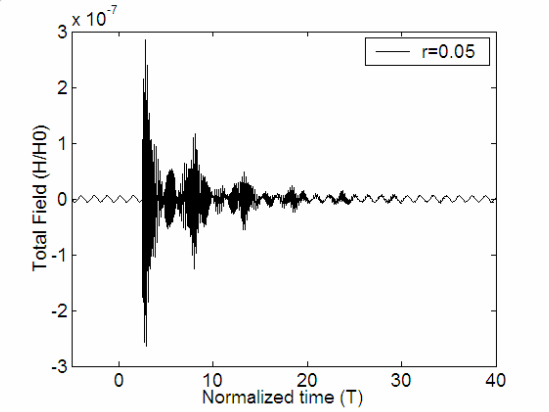


Fig. 1. Time dependence of the transformed field (r=0.05)

Abruptly decreasing the refractive index leads to an increase in the frequency, and increasing the refractive index leads to the opposite effect.

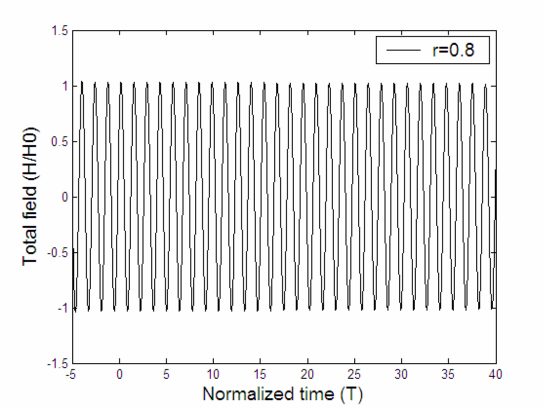


Fig. 2. Time dependence of the transformed field (r=0.8)

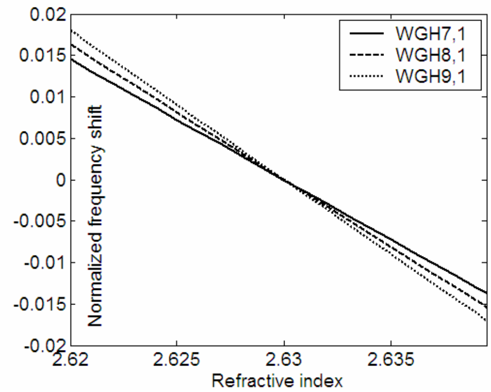


Fig. 3. Normalized frequency shift versus refractive index

Proposed approach has been extended to the case of a linear chain of coupled resonators. If resonators are brought togeter, four families of coupled modes with different types of symmetry with respect to Ox and Oy axes can be excited [10]. Numerical results are presented for two coupled modes with even-even (EE) and odd-odd (OO) symmetry along the Ox and Oy axes. Fig. 4 shows their near field patterns for N=6, with the $WGH_{8,1}$ mode considered as initial field (the same for Fig. 5, 6). Before zero moment of time, $n_1 = 2.63$.

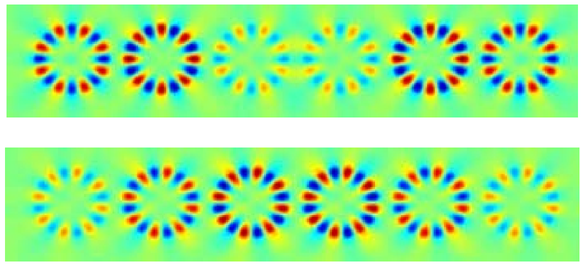

Fig. 4. Near field patterns of EE mode (above) and OO mode (below).

Fig. 5 represents the absolute value of the normalized frequency shift $\Delta w$ of the transformed mode versus the normalized separation distance between the resonators in the twin resonator structure (N=2). It is seen that for the distant resonator ( $d > a$ ) the frequency shift is the same for the OO and EE modes. The frequency shift decreases for the EE mode and increases for the OO coupled mode if the resonators are brought together.

If the number of resonators in the chain with small air-gaps gets larger, this leads to increase in the frequency shift for the OO coupled mode (Fig. 6).

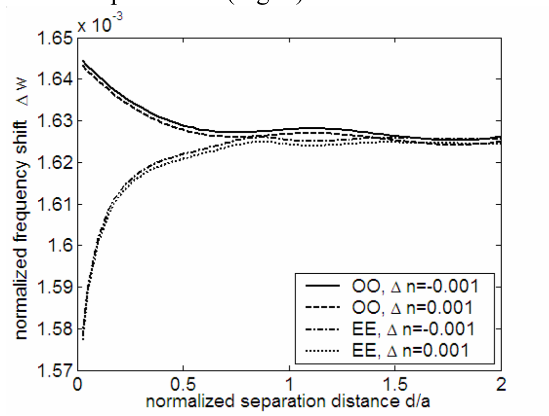


Fig. 5. The normalized frequency shift versus the normalized separation distance between the resonators, N=2

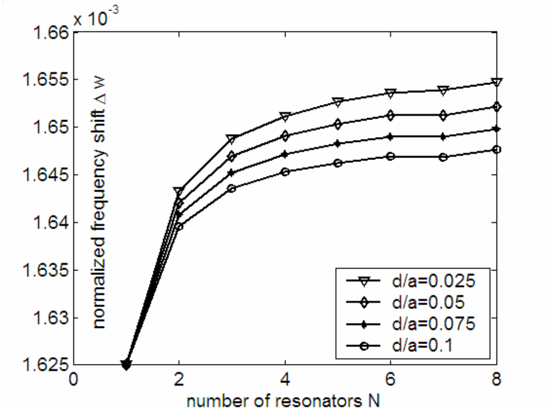


Fig. 6. The normalized frequency shift versus number of the resonators in the chain

## VI. Conclusion

In this paper, a theoretical analysis of WGM transformation due to the time variation of the permittivity in a single resonator and in a linear chain of coupled resonators has been developed. The theory is based on eigenfunction expansion in the Laplace transform domain and inversion of the solution into the time domain through the residue evaluation.

The obtained results reveal the resonant frequency shift of the transformed field. Enhancement of the frequency shift for the coupled modes with odd-odd symmetry in the chain has been shown.

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