

# Reduction of the measurand estimate bias for nonlinear model equation

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**Abstract.** The biases arising in the calculation of the measurand estimate for nonlinear measurement models are considered. The possibility of reducing these biases by applying the finite increments method is shown.

## 1. Errors of calculating the estimate of the measurand, caused by the nonlinearity of the model equation

The evaluation of the combined standard measurement uncertainty  $u_c(y)$  in GUM [1] is based on the law of propagation of uncertainty, in which the measurement model  $Y = f(X_1, X_2, \dots, X_N)$  (1) is represented by the first orders of the Taylor series expansion. The Note to clause 5.1.2 of the GUM states: “When the nonlinearity of  $f$  is significant, higher-order terms in the Taylor series expansion must be included in the expression for  $u_c^2(y)$ ”. In this case, the estimate  $y$  of the measurand  $Y$  is found by the formula:

$$y_{\text{GUM}} = f(x_1, x_2, \dots, x_N), \quad (2)$$

where  $x_1, x_2, \dots, x_N$  – are the estimates of the input quantities  $X_1, X_2, \dots, X_N$ .

Such an estimate will be biased in the case of nonlinear equation (1) and significant uncertainties of the input quantities [2].

Indeed, the expression (1) can be represented as a Taylor series expansion:

$$Y \approx f(x_1, x_2, \dots, x_N) + \sum_{j=1}^N \frac{\partial f}{\partial X_j} (X_j - x_j) + \frac{1}{2} \sum_{j,i=1}^N \left( \frac{\partial^2 f}{\partial X_j \partial X_i} \right) (X_j - x_j)(X_i - x_i) + R_0, \quad (3)$$

in which  $\frac{\partial f}{\partial X_j}$  – is the partial derivative (1) with respect to  $X_j$ ;  $\frac{\partial^2 f}{\partial X_j \partial X_i}$  – a second-order mixed derivative with respect to  $X_j$  and  $X_i$ ;  $R_0$  – remainder of the Taylor series.

The expectation of (3) if  $R_0 = 0$  will have the following form:

$$E(Y) = E[f(x_1, x_2, \dots, x_N)] + \sum_{j=1}^N \frac{\partial f}{\partial X_j} E(X_j - x_j) + \frac{1}{2} \sum_{j,i=1}^N \left( \frac{\partial^2 f}{\partial X_j \partial X_i} \right) E[(X_j - x_j)(X_i - x_i)]. \quad (4)$$

Since, based on the properties of the expectation,  $E(X_j - x_j) = 0$ , we obtain the equation:

$$E(Y) = f(x_1, x_2, \dots, x_N) + \frac{1}{2} \sum_{j,i=1}^N \left( \frac{\partial^2 f}{\partial X_j \partial X_i} \right) E[(X_j - x_j)(X_i - x_i)]. \quad (5)$$

Inasmuch as, for uncorrelated input quantities:

$$E[(X_j - x_j)(X_i - x_i)] = \begin{cases} E(X_j - x_j)^2 = u_j^2, & \text{when } j = i; \\ E(X_j - x_j)E(X_i - x_i) = 0, & \text{when } j \neq i, \end{cases}$$

where  $u_j$  – is the standard uncertainty of  $j$ -th input quantity, then:

$$E(Y) = f(x_1, x_2, \dots, x_N) + \frac{1}{2} \sum_{j=1}^N \left( \frac{\partial^2 f}{\partial X_j^2} \right) u_j^2. \quad (6)$$

The expression (6) includes the bias of the measurand numerical value estimate accepted in the GUM:

$$\Delta(y_{\text{GUM}}) = -\frac{1}{2} \sum_{j=1}^N \left( \frac{\partial^2 f}{\partial X_j^2} \right) u_j^2. \quad (7)$$

For function of one variable  $Y = f(X)$  the expression of bias has the following form:

$$\Delta(y_{\text{GUM}}) = -\frac{1}{2} \frac{\partial^2 f}{\partial x^2} u^2. \quad (8)$$

For example, for the power function  $Y = X^m$  the expression of bias will be equal to:

$$\Delta(y) = -\frac{m(m-1)}{2} x^{m-2} u^2 = -y \frac{m(m-1)}{2} \tilde{u}^2, \quad (9)$$

where  $\tilde{u}$  is the relative standard uncertainty of  $x$ :

$$\tilde{u} = \frac{u}{|x|}. \quad (10)$$

From the expression (9), we can obtain the expression for the relative bias:

$$\delta(y_{\text{GUM}}) = \frac{\Delta(y_{\text{GUM}})}{y_{\text{GUM}}} = -\frac{m(m-1)}{2} \tilde{u}^2. \quad (11)$$

The dependences  $\delta(y_{\text{GUM}}) = \varphi[\tilde{u}]$  for different powers  $m$  are presented in the Table 1 and in the Fig. 1. It is seen from Fig. 1, that when  $\tilde{u}(x) > 0,2$  using (2) leads to unreliable estimates of the measurand.

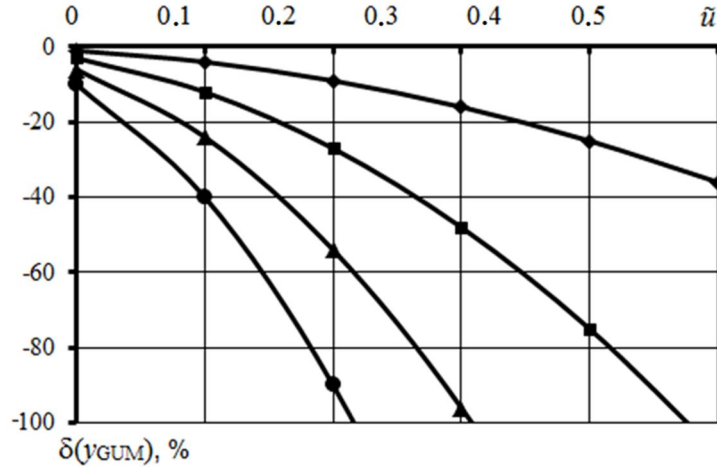


Fig.1 The dependences of the relative bias of measurand estimate on  $\tilde{u}$  for  
 ◆ –  $m=2$ ; ■ –  $m=3$ ; ▲ –  $m=4$ ; ● –  $m=5$ .

## 2. The application of the finite increment method

It is obvious that it is possible to reduce the bias of the measurand estimate for a nonlinear measurement model by taking into account the third term of the expansion in the Taylor series of the measurement model. To do this, we can use expression (7) as the correction to (2). However, (7) contains the second-order derivatives of the measurand, therefore, to simplify the solution of this problem, one can use the method of final increments (MFI) [2]. In this case, the second-order partial derivatives in expression (5) can be expressed in terms of the second-order difference derivatives for increments  $u_j$  [3]:

$$\frac{\partial^2 f}{\partial x_j^2} \approx \frac{f[x_1, \dots, (x_j + u_j), \dots, x_N] - 2f[x_1, \dots, x_N] + f[x_1, \dots, (x_j - u_j), \dots, x_N]}{u_j^2}. \quad (12)$$

Substituting expression (12) into (6), we obtain:

$$y_{\text{MFI}} = \sum_{j=1}^N \frac{f[x_1, \dots, (x_j + u_j), \dots, x_N] + f[x_1, \dots, (x_j - u_j), \dots, x_N]}{2} - (N-1)f[x_1, x_2, \dots, x_N]. \quad (13)$$

For a function of one variable  $Y = f(X)$ , the expression for  $y_{\text{MFI}}$  has the form:

$$y_{\text{MFI}} = \frac{f(x+u) + f(x-u)}{2}. \quad (14)$$

For example, for a power function  $Y = X^m$ , the expression for  $y_{\text{MFI}}$  is equal to:

$$y_{\text{MFI}} = \frac{(x+u)^m + (x-u)^m}{2}. \quad (15)$$

Taking into account the binomial Newton's formula:

$$(x+u)^m = \sum_{i=1}^m C_m^i x^{m-1} (\pm u)^i,$$

where

$$C_m^i = \frac{m!}{i!(m-i)!},$$

we obtain

$$\begin{aligned} y_{\text{MFI}} &= \frac{1}{2} \left[ \sum_{i=1}^m C_m^i x^{m-1} u^i + \sum_{i=1}^m C_m^i x^{m-1} (-u)^i \right] = x^m \left[ 1 + C_m^2 \tilde{u}^2 + C_m^4 \tilde{u}^4 + C_m^6 \tilde{u}^6 + \dots \right] = \\ &= x^m \left[ 1 + \frac{m(m-1)}{2} \tilde{u}^2 + \frac{m(m-1)(m-2)(m-3)}{24} \tilde{u}^4 + \frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{780} \tilde{u}^6 + \dots \right]. \quad (16) \end{aligned}$$

The expressions of  $y_{\text{MFI}}$  for different powers  $m$  are given in the Table 1.

Table 1. Expressions for different estimates of the measured quantity and relative bias for the power function

| $m$ | $y_{\text{GUM}}$ | $\delta(y_{\text{GUM}})$ | $E(X^m)$                                    | $y_{\text{MFI}}$                      | $\delta(y_{\text{MFI}})$                                       |
|-----|------------------|--------------------------|---|---------------------------------------|--|
| 1   | $x$              | 0                        | $x$   | $x$                                   | 0  |
| 2   | $x^2$            | $\tilde{u}^2$            | $x^2(1+\tilde{u}^2)$                        | $x^2(1+\tilde{u}^2)$                  | 0  |
| 3   | $x^3$            | $3\tilde{u}^2$           | $x^3(1+3\tilde{u}^2)$                       | $x^3(1+3\tilde{u}^2)$                 | 0  |
| 4   | $x^4$            | $6\tilde{u}^2$           | $x^4(1+6\tilde{u}^2 + \mu_4 \tilde{u}^4)$   | $x^4(1+6\tilde{u}^2 + \tilde{u}^4)$   | $\tilde{u}^4(1-\mu_4)/(1+6\tilde{u}^2 + \mu_4 \tilde{u}^4)$    |
| 5   | $x^5$            | $10\tilde{u}^2$          | $x^5(1+10\tilde{u}^2 + 5\mu_4 \tilde{u}^4)$ | $x^5(1+10\tilde{u}^2 + 5\tilde{u}^4)$ | $5\tilde{u}^4(1-\mu_4)/(1+10\tilde{u}^2 + 5\mu_4 \tilde{u}^4)$ |

The expressions for the expectation of the power function with account of the input quantity uncertainty were obtained by the method of moments [3] and are given in Table 1, where  $\mu_4$  is the normalized fourth-order central moment of the input quantity distribution law. For the normal law  $\mu_4 = 3.0$ ; for the uniform one  $\mu_4 = 1.8$ ; for the triangular one  $\mu_4 = 2.4$ ; for the U-shaped law  $\mu_4 = 1.5$ .

The dependences of the relative bias of the estimate  $y_{\text{MFI}}$  on  $\tilde{u}$  for  $m = 4$  and  $m = 5$  are shown in Fig. 2.

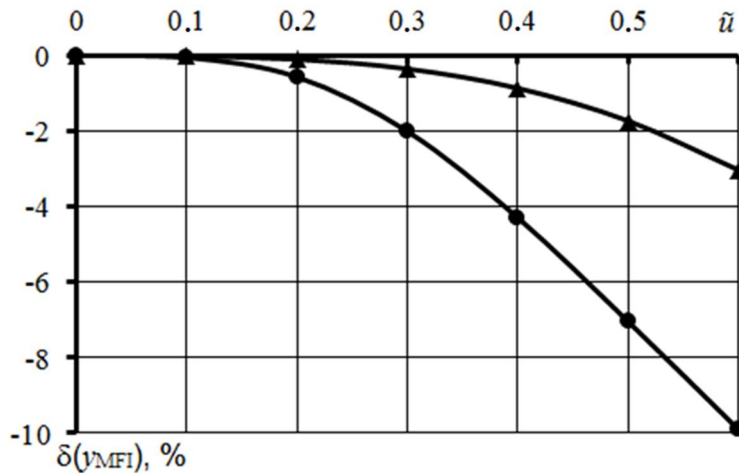


Fig. 2. The dependences of the relative bias of estimate  $y_{\text{MFI}}$  on  $\tilde{u}$  for  $\blacktriangle - m=4$ ;  $\bullet - m=5$ .

In Fig. 2 it is seen that the proposed method reduces the bias of the measurand estimate on 1-2 orders (for  $m=4$  and  $m=5$ ), and completely eliminates the bias for  $m=2$  and  $m=3$  (Table 1).

For a measurement model with two input quantities  $Y=f(X_1, X_2)$ , the expression for  $y_{\text{MFI}}$  has the form:

$$y_{\text{MFI}} = \frac{f[(x_1 + u_1), x_2] + f[(x_1 - u_1), x_2]}{2} + \frac{f[x_1, (x_2 + u_2)] + f[x_1, (x_2 - u_2)]}{2} - y_{\text{GUM}}. \quad (17)$$

The Table 2 shows the measurand estimates for different two variables functions and different values  $x_1$ ,  $x_2$  and standard uncertainties  $u_1$ ,  $u_2$  of the input quantities obtained with the help of GUM ( $y_{\text{GUM}}$ ), the MFI ( $y_{\text{MFI}}$ ) and the Monte Carlo method ( $y_{\text{MCM}}$ ) for normal and uniform distribution laws.

Table 2. The values of measurand estimates for different measurement model

| $Y=f(X_1, X_2)$            | $x_1$ | $x_2$ | $u_1$ | $u_2$ | $y_{\text{GUM}}$ | $y_{\text{MFI}}$ | $y_{\text{MCM}}$<br>normal | $y_{\text{MCM}}$<br>uniform |
|----------------------------|-------|-------|-------|-------|------------------|------------------|----------------------------|-----------------------------|
| $Y = X_1^2 + X_2^2$        | 0     | 0     | 1     | 1     | 0                | 2                | 2                          | 2                           |
|                            | 1     | 1     | 1     | 1     | 2                | 4                | 4                          | 4                           |
|                            | 1     | 1     | 0.5   | 0.5   | 2                | 2.5              | 2.5                        | 2.5                         |
| $Y = X_1^4 + X_2^4$        | 0     | 0     | 1     | 1     | 0                | 2                | 6                          | 3.6                         |
|                            | 1     | 1     | 1     | 1     | 2                | 16               | 20                         | 17.6                        |
|                            | 1     | 1     | 0.5   | 0.5   | 2                | 5.13             | 5.38                       | 5.23                        |
| $Y = \sqrt{X_1^2 + X_2^2}$ | 0     | 0     | 1     | 1     | 0                | 2                | 1.25                       | 1.33                        |
|                            | 1     | 1     | 1     | 1     | 1.41             | 1.82             | 1.81                       | 1.81                        |
|                            | 1     | 1     | 0.5   | 0.5   | 1.41             | 1.51             | 1.51                       | 1.5                         |
| $Y = (X_1^2 + X_2^2)^2$    | 0     | 0     | 1     | 1     | 0                | 2                | 8                          | 5.6                         |
|                            | 1     | 1     | 1     | 1     | 4                | 22               | 28                         | 26                          |
|                            | 1     | 1     | 0.5   | 0.5   | 4                | 8.13             | 8.5                        | 8.3                         |

In subclause 9.4 of Supplement 1 to the GUM [4], an example of ‘‘Comparison loss in the calibration of a microwave power meter’’ is considered for the function  $Y = X_1^2 + X_2^2$  and for  $x_1 = 0$ ,  $x_2 = 0$  and standard uncertainties  $u_1 = 0.005$ ,  $u_2 = 0.005$ . In this case we have:

$$\begin{aligned}
 y_{\text{GUM}} &= 0+0=0. \\
 y_{\text{MFI}} &= \frac{[(x_1 + u_1)^2 + x_2^2] + [(x_1 - u_1)^2 + x_2^2]}{2} + \frac{[x_1^2 + (x_2 + u_2)^2] + [x_1^2 + (x_2 - u_2)^2]}{2} - (x_1^2 + x_2^2) = \\
 &= \frac{[(0 + u_1)^2 + 0^2] + [(0 - u_1)^2 + 0^2]}{2} + \frac{[0^2 + (0 + u_2)^2] + [0^2 + (0 - u_2)^2]}{2} - 0 = \\
 &= u_1^2 + u_2^2 = 0.005^2 + 0.005^2 = 0.00005.
 \end{aligned}$$

In [4] by the Monte Carlo method the value  $y_{\text{MCM}} = 0.00005$  has been obtained, which coincides with the obtained value  $y_{\text{MFI}}$ .

Thus, the proposed approach makes it possible to reduce the bias in the measurand estimate for nonlinear model equations.

## References

- [1] Evaluation of measurement data – Guide to the Expression of Uncertainty in Measurement: JCGM 100:2008. – JCGM, 2008. – 120 p.
- [2] Botsiura O.A. Application of the method of final increments for the estimation of the results of indirect measurements // Standardization, certification, quality. 2017. No. 1. Pp. 57-65 (in Ukrainian).
- [3] Amosov A.A., Dubinsky Yu.A., Kopchenov N.V. Computational Methods for Engineers: Tutorial. Moscow: Higher school, 1994. – 544 p. (in Russian).  
Evaluation of measurement data – Supplement 1 to the ‘‘Guide to the expression of uncertainty in measurement’’ – Propagation of distributions using a Monte Carlo method: JCGM 101:2008, 2008. – 90 p.