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STRUCTURE OF FACTORIZATION PROCESSES

A granular computing conception can be interpreted as information analysis of both as a group of elements associated with internal or external properties of arbitrary nature data. Multialgebraic systems as a mathematical tool for quotient sets processing is one of possible approaches for granule and granule structure synthesis and analysis. Embedded (implicit, induced by data nature) and external (given, required for the application domain) regularities can be partitioned in different modes so such factorization processes are of interest for automatic information roughening, or on the contrary, it detailing.

GRANULATION, QUOTIENT SET, N-ARY RELATION, DIFUNCTIONALITY

Introduction

The genesis of searching for sufficient (ideally, necessary) level of data (information, knowledge) presentation accuracy lies in the compromise region, as inconsistency of criteria is one of the key factors determining correctness and validity of results. A paradigm of granular computing consists in grouping elements together (in a granule) by indistinguishability, similarity, proximity or functionality in arbitrary feature or signal spaces [1]. Taking into account a semantic interpretation of why two objects are put into the same granule and how two objects are related with each other it provides one of a general methodology for intelligent data analysis on different levels of roughening or detailing. Internal, external and contextual properties of granules, collective structure of a family of granules and hierarchical structure of granules represent a possible foundation for qualitative/quantitative characterization of levels of abstraction, detail, control, explanation, difficulty, organization and so on [1]. Because the quotient spaces have a set of favorable structural properties they become a suitable model of granulation [2, 3].

Multialgebraic systems [4-6] can be pointed out as one of the possible granular computing 'languages'. They generalize results of traditional algebraic systems [7] and allow synthesizing and identifying algebras, models, and algebraic systems (with carriers that join families of sets of arbitrary nature), for different mathematical structures, eventually allowing operation with granules as equivalence classes without distinguishing particular class members.

1. Problem Statement

It is clear that an arbitrary n-ary relation induces any k-ary relation from 1 to n on different carriers. This is done as follows: by assuming that n-ary relation is initially defined on a carrier in a form of Cartesian product $\overline{A} = A_1 \times \ldots \times A_n$ of sets of arbitrary nature (in general case), any ordered and examined collection of sets $\{A_1,\ldots,A_n\}$ produces Cartesian product $B_1{=}A_{k_1} \times \ldots \times A_{k_s}$ which is a set. Complement $\{A_{k_1},\ldots,A_{k_s}\}$ to $\{A_1,\ldots,A_n\}$ in a form of Cartesian product produces set $B_2{=}A_{r_1} \times A_{r_2} \times \ldots \times A_{r_l}$, thus, in whole, binary relation on a carrier $B_1 \times B_2$ is induced.

Ternary (or any other) relation can be induced similarly. Thereby initial n-ary relation S may induce k-ary relation S/P_k which amounts to a certain partition P_k of set $\{A_1,\ldots,A_n\}$ into k ordered parts. The goal of the paper consists in investigation of conditions which provide various partitions produced by relation subsets.

2. General Scheme of Multialgebraic Structure Induction

Relation S/P_k is formally specified on $B_1 \times \ldots \times B_k$ where

$$\begin{aligned} &B_1{=}A_{r_{11}} \times A_{r_{12}} \times \ldots \times A_{r_{1l_1}}, \\ &B_2{=}A_{r_{21}} \times A_{r_{22}} \times \ldots \times A_{r_{2l_2}}, \\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ &B_k{=}A_{r_{k1}} \times A_{r_{k2}} \times \ldots \times A_{r_{kl_k}} \end{aligned} \qquad (1)$$

and

$$\begin{aligned} &r_{11}<r_{12}<\ldots<r_{1l_1}, \\ &r_{21}<r_{22}<\ldots<r_{2l_2}, \\ &\quad . \quad . \quad . \quad . \quad . \\ &r_{k1}<r_{k2}<\ldots<r_{kl_k} \end{aligned}$$

with this, sets from the right side of the equations (1) belong to collection of sets $\{A_1,\ldots,A_n\}$, and $l_1+l_2+\ldots+l_k=n$.

It is not hard to notice that relation S/P_k is specified on a collection of points, each of which is a result of projection of $\overline{A}$ into B_i, $i=\overline{1,k}$. Indeed, by fixing element $\overline{a}=(a_1,\ldots,a_n) \in \overline{A}$, where $a_i \in A_i$, $i=\overline{1,n}$, and elements $\overline{b}_1,\ldots,\overline{b}_k$ respectively, for which $\overline{b}_i \in B_i$, $i=\overline{1,k}$ and

$$\begin{aligned} &\overline{b}_1=(a_{r_{11}},a_{r_{12}},\ldots,a_{r_{1l_1}}), \\ &\overline{b}_2=(a_{r_{21}},a_{r_{22}},\ldots,a_{r_{2l_2}}), \\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ &\overline{b}_k=(a_{r_{k1}},a_{r_{k2}},\ldots,a_{r_{kl_k}}) \end{aligned}$$

are held, then mappings

$$f_i(\overline{a})=\overline{b}_i, \; i=\overline{1,k}$$

are simply projectors, as is customary for arithmetical spaces $\mathbb{R}^n$. When we pick some ordered set of coordinates S from coordinates n, we actually project $\mathbb{R}^n$ into $\mathbb{R}^S$. It can be schematically presented as follows.

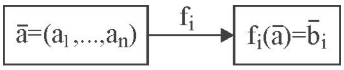

Fig. 1

Now consider induced relation S/P_k specified on $\bar{B}=B_1\times\ldots\times B_k$. As for any relation, equivalence classes will be formed on each B_i in a known manner, i.e. $\bar{b}_i\sim\bar{b}_i{}'$ if

$$S/P_k(\bar{b}_1,\bar{b}_2,\ldots,\bar{b}_{i-1},\bar{b}_i,\bar{b}_{i+1},\ldots,\bar{b}_k)\equiv$$
$$\equiv S/P_k(\bar{b}_1,\bar{b}_2,\ldots,\bar{b}_{i-1},\bar{b}_i{}',\bar{b}_{i+1},\ldots,\bar{b}_k)$$

for any element

$$(\bar{b}_1,\ldots,\bar{b}_{i-1},\bar{b}_i,\bar{b}_{i+1},\ldots,\bar{b}_k)\in$$
$$\in B_1\times\ldots\times B_{i-1}\times B_i\times B_{i+1}\times\ldots\times B_k.$$

Actually, indiscernible elements (from the point of the relation) fall into one equivalence class. Factorization procedure takes place at this stage, and k-ary multirelation $M[S/P_k]$ specified on equivalence classes is induced along with corresponding mappings that are to be called classifying mappings and denoted by α_i. It can be schematically illustrated as follows (see fig. 2).

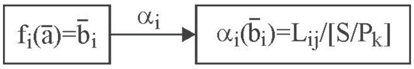

Fig. 2

Here, $j=\overline{1,l_i}$, $L_{ij}/[S/P_k]$ denotes equivalence classes induced on set B_i, and l_i denotes the amount of these classes. Thus, the following set is formed

$$L_i=\{L_{i1}/[S/P_k],\ldots,L_{il_i}/[S/P_k]\}$$

and $i=\overline{1,k}$. Eventually, k-ary multirelation $M[S/P_k]$ is specified on Cartesian product $L_1\times\ldots\times L_k$.

Now it is easy to see that the general scheme of induction of k-ary multirelation $M[S/P_k]$ by arbitrary n-ary relation S looks like this (see fig. 3).

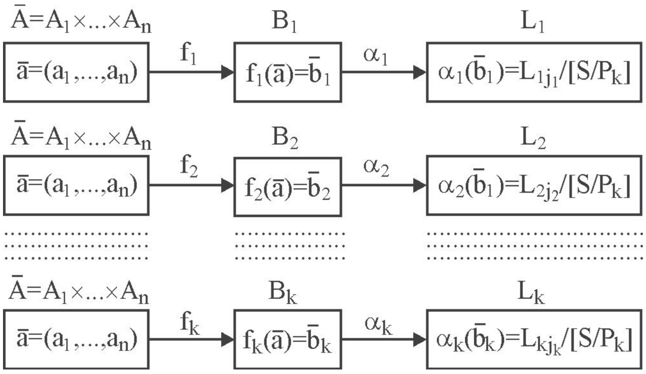

Fig. 3

In fact, if the value of initial n-ary relation S at point $\bar{a}$ is known, then the value of k-ary multirelation $M[S/P_k]$ is formed according to the scheme from fig. 3:

– at the first stage, the value of k-ary relation S/P_k is defined by the following equation brought on by projectors $\{f_i\}_{i=1}^k$

$$S(\bar{a})=S/P_k(f_1(\bar{a}),\ldots,f_k(\bar{a}))=S/P_k(\bar{b}_1,\ldots,\bar{b}_k) \quad (2)$$

– at the second stage, the value of k-ary multirelation $M[S/P_k]$ is defined by the following equation brought on by classifying mappings $\{\alpha_i\}_{i=1}^k$

$$S/P_k(\bar{b}_1,\ldots,\bar{b}_k)=M[S/P_k](\alpha_1(\bar{b}_1),\ldots,\alpha_k(\bar{b}_k)) \quad (3)$$

From (2) and (3) the whole chain of equations is obtained

$$S(a_1,\ldots,a_n)=S(\bar{a})=S/P_k(f_1(\bar{a}),\ldots,f_k(\bar{a}))=$$
$$=S/P_k(\bar{b}_1,\ldots,\bar{b}_k)=M[S/P_k](\alpha_1(\bar{b}_1),\ldots,\alpha_k(\bar{b}_k))$$

which explicitly shows relationship between n-ary relation S and its induced k-ary multirelation $M[S/P_k]$.

For now, let us make some remarks.

Remark 1. General induction procedure of k-ary multirelation $M[S/P_k]$ by arbitrary n-ary relation S always takes place.

Next lemma essentially follows from the previous remark in terms of algebraic system terminology [7].

Lemma 1. Arbitrary model $\langle\bar{A},S\rangle$, where S is n-ary relation with carrier $\bar{A}=A_1\times\ldots\times A_n$, always produce model $\langle\bar{L},M[S/P_k]\rangle$ where $M[S/P_k]$ is k-ary relation (or, more precisely, multirelation) with carrier $\bar{L}=L_1\times\ldots\times L_k$, while L_i is a set of equivalence classes with $i=\overline{1,k}$, $k=\overline{1,n}$.

Remark 2. In general, sets L_i may be of different cardinality, but algebraic structures of type $\langle A,\mathcal{F},\mathcal{P}\rangle$ are met more often, where A is an arbitrary set (carrier), $\mathcal{F}$ is a set of relations defined on Cartesian powers of A with different order, and $\mathcal{P}$ is a set of operations defined on Cartesian powers of A with different order. Well-known algebraic structures (such as semigroup, group, ring, field, etc.) fall into this scheme. Hence the next remark arises.

Remark 3. It is worth to mention the following situations for the general scheme of factorization and induction of multisystems:

i) n-ary relation induces k-ary multirelation with a common carrier, i.e. in a form of Cartesian power of order k;

ii) n-ary relation induces a set of multirelations of different order, but with a carrier common for all of them in a form of Cartesian powers of different order;

iii) a set of relations of different order induces a set of multirelations of different order, but with a carrier common for all of them.

For instance, by considering previous works [8, 9], multigroup falls into situation described in item i), which is the most simple one and should be examined at the beginning.

Let us state a problem to find out conditions under which k-ary multirelation $M[S/P_k]$, induced by n-ary relation S, has a common carrier.

First, it should be noted that in this case, the sets of equivalence classes $L_1,L_2,\ldots,L_k$ must be equipotent

$$|L_1|=|L_2|=\ldots=|L_k|.$$

This means that with an accuracy of nature of the sets, there is a set L and some one-to-one mapping φ for which the following is true

$$\varphi(L_i) = L,\ i = \overline{1,k}.$$

In such a case, the general scheme of induction of k-ary multirelation $M[S/P_k]$ by arbitrary n-ary relation S, given in fig. 3, should be refined as each of its rows is expanded by an element with the following scheme (fig. 4).

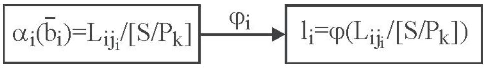

Fig. 4

It means that the whole scheme will look as follows (fig. 5).

Here, mappings f_i are projectors, α_i are called classifying mappings, one-to-one mapping φ is added, which will be called resulting mapping in future.

It should be emphasized that a whole set of mappings appears as well in the scheme from fig. 5. They are going to play an important role in future, so they should be observed more carefully.

Consider two arbitrary elements $\bar{a}_1$ and $\bar{a}_2$ that belong to $\bar{A}$, and fix two projectors f_p and f_q, $p,q=\overline{1,k}$. It is clear that after the influence of classifying mappings α_p and α_q on elements $f_p(\bar{a}_1)$ and $f_q(\bar{a}_2)$, the latest fall into some equivalence classes that belong to sets L_p and L_q respectively, i.e. the following superpositions of projectors and classifying mappings appear

$$[\alpha_p \circ f_p](\bar{a}_1) = \alpha_p(f_p(\bar{a}_1)) = L_{pj_p}/[S/P_k],$$
$$[\alpha_q \circ f_q](\bar{a}_2) = \alpha_q(f_q(\bar{a}_2)) = L_{qj_q}/[S/P_k]$$

where symbol '$\circ$' indicates superposition of mappings.

Assume that the following is fulfilled under effect of final one-to-one mapping φ

$$\varphi(L_{pj_p}/[S/P_k]) = \varphi(L_{qj_q}/[S/P_k]),$$

i.e. we fall into the same class or element of set L. Hence, elements $\bar{a}_1$ and $\bar{a}_2$ from set $\bar{A}$ may be put into correspondence with each other due to belonging to the same class or element of set L. The next given equations are true for these elements

$$[\varphi \circ \alpha_p \circ f_p](\bar{a}_1) = [\varphi \circ \alpha_q \circ f_q](\bar{a}_2) \quad (4)$$

or

$$\varphi\left(\alpha_p\left(f_p(\bar{a}_1)\right)\right) = \varphi\left(\alpha_q\left(f_q(\bar{a}_2)\right)\right). \quad (5)$$

In such a case the next rule can define mapping ψ_{pq} which acts from $\bar{A}$ to $\bar{A}$ or $\psi_{pq}: \bar{A} \to \bar{A}$

$$\varphi\left(\alpha_p\left(f_p(\bar{a}_1)\right)\right) = \varphi\left(\alpha_q\left(f_q(\bar{a}_2)\right)\right)$$

if only (4) or (5) is held.

It will be called classifying mapping on $\bar{A}^2$.

Examine some properties of these mappings. Denote superposition of mappings φ_p, α_p, f_p by F_p, i.e.

$$F_p = \varphi_p \circ \alpha_p \circ f_p. \quad (6)$$

From the rule (6) it follows that

$$\psi_{pq}(\bar{a}_1) = \bar{a}_2 \Leftrightarrow F_p(\bar{a}_1) = F_q(\bar{a}_2). \quad (7)$$

When $p = q$, the following is obtained

$$\psi_{pp}(\overline{a_1}) = \overline{a_2} \Leftrightarrow F_p(\overline{a_1}) = F_p(\overline{a_2}). \quad (8)$$

From (8) it is clear that lines from a level of functional mapping F_p (mappings F_p, φ_p, α_p and f_p are functional mappings with a single image, i.e. for F_p and φ_p there is a single element of set L, f_p and α_p are the sole elements of sets B_p and L_p correspondingly) are an image and pre-image of mapping ψ_{pp} for any $p=\overline{1,k}$, or they specify partition on $\bar{A}$ and, hence, an equivalence on $\bar{A}$ in a following way

$$\bar{a}_1 \sim \bar{a}_2 \Leftrightarrow \psi_{pp}(\bar{a}_1) = \bar{a}_2.$$

When $p \neq q$, mapping ψ_{pq} provides correspondence between two partitions on $\bar{A}$, that are induced by lines from a level of mappings F_p and F_q.

Formally it is expressed by equations (7) and (8), but in fact it means that mappings ψ_{pp} and ψ_{pq} are multivalued, and mapping ψ_{pp} provides correspondence for any $\bar{a}_1$ from $\bar{A}$ with any element $\bar{a}_2$ from $\bar{A}$, if they fall into the same class (element of set L) under effect of mapping F_p (they have one image). This can be schematically presented as follows i.e. correspondence for any element $\bar{a}_1$ from set $\bar{A}_s$ (partition element or a line of level F_p) under mapping ψ_{pp} is provided with any element $\bar{a}_2$ from this set.

If mapping ψ_{pq} is taken into consideration, there are two partitions for it, namely lines from a level of mapping F_p (see fig. 6) and lines of level F_q (see fig. 8)

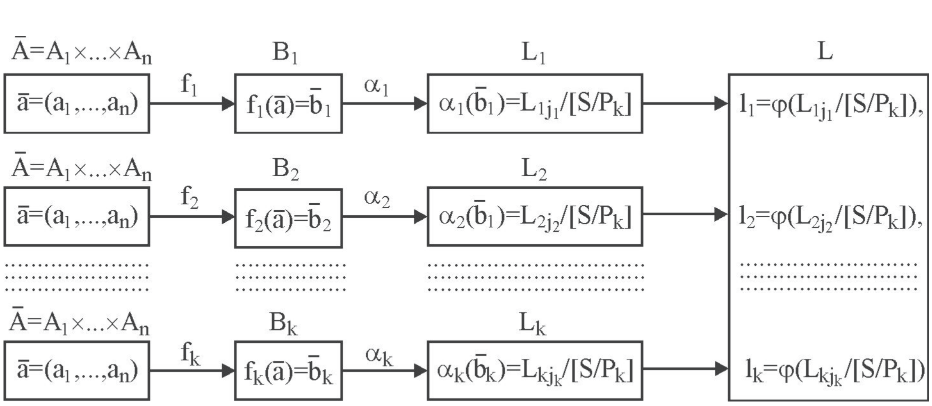

Fig. 5

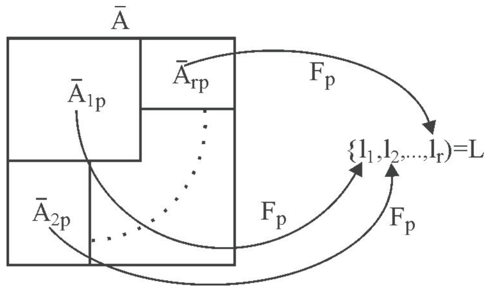

Fig. 6

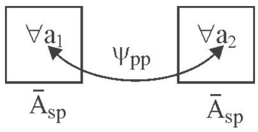

Fig. 7

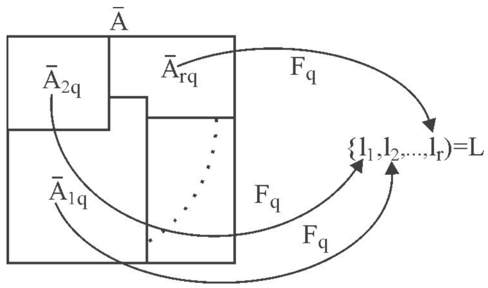

Fig. 8

i.e. if set $\bar{A}_{s_1p}$ (line from a level of mapping F_p) under this mapping F_p moves to the same class (element of set L) as a set $\bar{A}_{s_2q}$ (line of level F_q) under effect of mapping F_q (see fig. 10), then the scheme from fig. 9 takes place: mapping ψ_{pq} provides correspondence for any element $\bar{a}_1$ from set $\bar{A}_{s_1p}$ with any element $\bar{a}_2$ from set $\bar{A}_{s_2q}$. Thus, ψ_{pq} is multivalued mappings between elements that move to the same element of set L under mappings F_p and F_q.

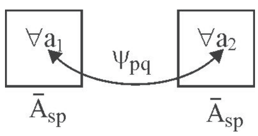

Fig. 9

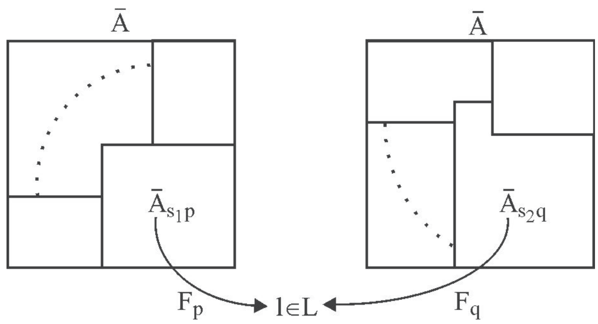

Fig. 10

Remark 4. In the figures above cardinality of set L is defined as a finite one for simplicity, though it is not obvious.

It is not hard notice that lines from a level of mappings F_p, notably set $\bar{A}_{s_1p}$ with any p, form a collection or a set which cardinality does not depend on p. Thus, by fixing $p \neq q$, binary relation T_{pq} defined on Cartesian square $\bar{A}^2$ and built under the following rule

$$T_{pq}(\bar{a}_1, \bar{a}_2) = 1 \Leftrightarrow F_p(\bar{a}_1) = F_q(\bar{a}_2) \qquad (9)$$

will be the relation of difunctionality.

Indeed, equations that are held for the relation T_{pq}

$$T_{pq}(\bar{a}_1, \bar{a}_2) = 1, T_{pq}(\bar{a}_3, \bar{a}_2) = 1, T_{pq}(\bar{a}_3, \bar{a}_4) = 1 \qquad (10)$$

influence a number of equations for mappings F_p and F_q

$$F_p(\bar{a}_1) = F_q(\bar{a}_2), F_p(\bar{a}_3) = F_q(\bar{a}_2), F_p(\bar{a}_3) = F_q(\bar{a}_4),$$

from which it is clear that

$$F_p(\bar{a}_1) = F_q(\bar{a}_4),$$

and because of (9), the following equation is obtained

$$T_{pq}(\bar{a}_1, \bar{a}_4) = 1. \qquad (11)$$

Thus, for the relation T_{pq} equation (11) is obtained from a number of equations in (10), which testifies difunctionality of the relation T_{pq}.

In general, steps needed to be performed for factorization procedure or induction of k-ary relation with a common carrier using mappings F_p look simple. If mappings F_p have a common image, i.e.

$$\text{Im } F_p = L, \forall p \in \{1, ..., k\},$$

it means that acting projector at the beginning, and then the classifier, superposition of which is the mapping F_p, influences induction of binary relations T_{pq}, all of which are difunctional in this case.

Thus, next lemma is proved.

Lemma 2. An arbitrary model $\langle \bar{A}, S \rangle$, where S is n-ary relation with carrier $\bar{A} = A_1 \times A_2 \times ... \times A_n$, produces model $\langle \bar{L}, M[S/P_k] \rangle$ where $M[S/P_k]$ is k-ary multirelation with carrier $\bar{L} = L^k$ (common carrier), then in such a case binary relations T_{pq} are difunctionality relations under $\forall p, q \in \{1, ..., k\}$

Remark 5. If $p=q$ then relation T_{pq} is an equivalence relation that is a special case of difunctional relation.

Consequently, difunctionality of the relations T_{pq} is a necessary condition for common carrier formation. A question arises concerning sufficient conditions.

Conclusion

Within the scope of granular computing conception on the base of multialgebraic system tools a problem of arbitrary arity relation reduction has been considered. This reduction produces different quotient sets which represent unified carriers for various multialgebraic structures. A necessary condition for common carrier formation has been established and search for sufficient conditions remains a unsolved problem.

References: 1. Handbook of granular computing / Pedrycz W., Skowron A., Kreinovich V. (eds.). – Chichester: John Wiley and Sons Ltd., 2008. – 1116 p. **2.** *Yao, Y.Y.* A partition model of granular computing. // Transactions on Rough Sets I / Peters

J.F., Skowron A., Grzymala-Busse J.W., et al. (eds.). Lecture Notes in Computer Science. – Berlin, Heidelberg: Springer-Verlag. – Vol. 3100. – 2004. – pp. 232-253. **3.** *Zhang L., Zhang B.* The quotient space theory of problem solving // Fundamenta Informaticae. – Vol. 59, No. 2-3. – 2004. – pp. 287-298. **4.** *Mashtalir V.P., Shlyakhov V.V.* Properties of multialgebraic systems in problems of comparative recognition // Cybernetics and Systems Analysis. – Vol. 39, No. 6. – 2003. – pp. 790–804. **5.** *Kagramanyan A., Mashtalir V., Shlyakhov V.* Multialgebraic systems in information granulation // International Journal "Information Theories and Applications" – 2008. – Vol. 15., No 1. – pp. 55-63. **6.** *Kagramanyan A., Mashtalir V., Shlyakhov V.* Multialgebraic structures existence for granular computing // Proc. of. XIII-th International Conference Knowledge-Dialogue-Solutions. – Sofia: FOI-COMMERCE. – Vol. I. – 2007. – pp. 322-330. **7.** *Maltsev A.I.* Algebraic systems. – Moskow: Nauka, 1970. – 392 p. (in Russian) **8.** *Mashtalir V., Mikhnova E., Shlyakhov V.* Multigroups in Granular Computing Paradigm. Part 1. Formalistic Premises // Proceedings of 19-th Zittau East-West Fuzzy Colloquium. – Zittau: HS Zittau / Görlitz. – 2012. – pp. 226–233. **9.** *Kinoshenko Ie., Mashtalir S., Shlyakhov V.* Multigroups in granular computing paradigm. Part 2. Conditions of existence and examples // Proceedings of 19-th Zittau East-West Fuzzy Colloquium. – Zittau: HS Zittau/Görlitz. – 2012. – pp. 234–241.

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Структура факторизационных процессов / А. Каграманян // Бионика интеллекта: научн.-техн. журнал. – 2014. – № 1 (82). – С. 33–37.

Концепция грануляционного исчисления может трактоваться как анализ информации в виде групп элементов, связанных внутренними или внешними свойствами данных произвольной природы. Мультиалгебраические системы как математический аппарат для обработки фактор-множеств – один из возможных подходов к синтезу и анализу гранул и их структур. Внедренные (неявные, индуцированные природой данных) и внешние (заданные, требуемые прикладными задачами) взаимосвязи могут разбиваться различными путями, поэтому эти факторизационные процессы интересны для автоматического огрубления или, напротив, детализации информации.

Ил.10. Библиогр.: 9 назв.

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Структура факторизаційних процесів / О. Каграманян // Біоніка інтелекту: наук.-техн. журнал. – 2014. – № 1 (82). – С. 33–37.

Концепція грануляційних обчислень трактуватися як аналіз інформації у вигляді груп елементів, які пов'язані внутрішніми або зовнішніми властивостями даних довільної природи. Мультіалгебраїчні системи як математичний апарат до синтезу та аналізу гранул та їх структур. Впроваджені (неявні, індуковані природою даних) та зовнішні (задані, які необхідні прикладними задачами) взаємозв'язки можуть розбиватися різними шляхами, тому ці факторизаційні процеси цікаві для автоматичного огрубіння або, навпаки, деталізації інформації.

Іл. 10. Бібліогр.: 9 найм.