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NUMERICAL ANALYSIS OF THE PROBLEM OF FLOW PAST A CYLINDRICAL BODY APPLYING THE R-FUNCTIONS METHOD AND THE GALERKIN METHOD

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Abstract. The article considers the stationary problem of viscous incompressible fluid flow past a cylindrical body. For solving the problem it is proposed a numerical method, based on the joint use of R-functions method and the Galerkin method. The computational experiment has been conducted for the task of flow past square cylinder for different Reynolds numbers.

Key words: viscous incompressible fluid, flow problem, the R-functions method, the Galerkin method.

INTRODUCTION

Recently, mathematical modeling and numerical analysis are increasingly used to study the dynamics of a viscous fluid. The necessity to simulate viscous flows occurs, for example, in fluid dynamics, thermal power, chemical kinetics, biomedicine, radio electronics, etc. [2-4, 11, 20].

The Navier-Stokes equations [12, 16, 17], describing such problems, have a number of specific features, such as nonlinearity and the presence of a small parameter at the highest derivative (the value is inverse to the Reynolds number). Furthermore, they often have to be solved in the areas of complicated geometry. When solving the exterior problems, the area under consideration may also be unlimited, but in the numerical solution it is modeled as a finite domain. It complicates the implementation of boundary conditions at infinity. Moreover, conditions at infinity are being demolished on certain contour, located far enough away from the streamlined body, which leads to additional errors in the approximate solution.

There is an extensive class of flows in which the nonlinear terms can be neglected to obtain the linear problem. The complete neglect of the inertial terms leads to the so-called equations of creeping flows or Stokes equations [5,11,13]. But there is no solution of the Stokes equations (Stokes paradox [5,11,25,26]) for the problem of unbounded viscous incompressible fluid flow past a cylindrical body. In this case, the Oseen approximation should be used [1,11,25].

When solving the hydrodynamics problems, the adequate consideration of the area geometry is important. It is implemented in a variety of computational methods with different degrees of effectiveness. The constructive apparatus of the R-functions theory of V.L. Rvachev [9,22-24], Ukraine National Academy of Science academician, allows to take into account the geometry of the area accurately and satisfy the boundary conditions of the problem precisely. The R-functions method in hydrodynamics problems was used by Kolosova S.V., Suvorova I.G., Maksimenko-Sheiko K.V., Sidorov M.V., but the problems of calculating the ideal fluid flows [6] and the viscous fluid flows in limited areas [8,28-30] or in the presence of helical symmetry [18] were considered.

The problems of external viscous fluid flows around bodies using the R-functions method were discussed in [7,13-15]. In [13] the problem of calculating the external slow viscous incompressible fluid flows past bodies in spherical and cylindrical coordinate systems was studied. For solving the nonlinear stationary problem of the viscous incompressible fluid flow past the cylindrical body in a cylindrical coordinate system, the paper [14] proposed a numerical method, based on the joint use of the R-functions method, the successive approximation method and the Galerkin-Petrov method. In [7] the application of the R-functions method, the successive approximation and the Galerkin-Petrov methods to the calculation of axisymmetric viscous incompressible fluid flows (the flowing around the finite bodies of rotation) was considered. In [15] the problem of mass transfer of the body of revolution with uniform translational flow was considered.

The purpose of this research is to apply the R-functions method and the Galerkin method for mathematical modeling of the linear and nonlinear stationary problem of viscous incompressible fluid flow past the cylindrical body in a rectangular coordinate system.

PROBLEM STATEMENT

Problem 1. Let us consider the problem of slow uniform viscous incompressible fluid flow with velocity U_∞ past a cylindrical body, the cross-section of which is a finite region Ω with piecewise continuous boundary $\partial\Omega$ [1,11,19]:

$$\Delta^2\psi + \mathrm{Re}\cdot A(\Delta\psi) = 0 \text{ outside } \overline{\Omega}, \quad (1)$$

$$\psi\big|_{\partial\Omega} = 0, \ \left.\frac{\partial\psi}{\partial\mathbf{n}}\right|_{\partial\Omega} = 0, \quad (2)$$

$$\psi \sim U_\infty y \text{ as } \sqrt{x^2+y^2} \to \infty, \quad (3)$$

where:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4},$$

$A(\Delta\psi) = -\dfrac{\partial\Delta\psi}{\partial x}$, Re is the Reynolds number, $\psi = \psi(x,y)$ is the stream function, associated with the velocity vector components by means of these relations:

$$v_x = \frac{\partial\psi}{\partial y}, \ v_y = -\frac{\partial\psi}{\partial x},$$

$\mathbf{n}$ is external normal to the $\partial\Omega$.

Problem 2. Consider the nonlinear stationary problem of viscous incompressible fluid flow past a cylindrical body [21]. In this case, the stream function satisfies the equation:

$$\Delta^2\psi = \mathrm{Re}\left(\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}\right) \text{ outside } \overline{\Omega}. \quad (4)$$

Eq. (4) is supplemented by the boundary conditions (2) and the condition at infinity (3).

NUMERICAL METHOD

The R-functions method [9,22-24] of V.L. Rvachev, Ukraine National Academy of Science academician, is proposed for solving the problem 1 and 2.

Let outside of $\overline{\Omega}$ a sufficiently smooth function $\omega(x,y)$, that satisfies the following properties, is known:

$$1)\ \omega(x,y) > 0 \text{ outside } \overline{\Omega},$$
$$2)\ \omega(x,y)\big|_{\partial\Omega} = 0, \quad (5)$$
$$3)\ \left.\frac{\partial\omega}{\partial\mathbf{n}}\right|_{\partial\Omega} = -1,$$

where: $\mathbf{n}$ is external normal to the $\partial\Omega$.

Let us introduce a sufficiently smooth function $y = f_M(x)$ from [27], which satisfies the following requirements:

$$\text{a) } f_M(0) = 0, \quad \text{b) } f'_M(0) = 1,$$
$$\text{c) } f'_M(0) \ge 0 \ \forall x \ge 0, \quad (6)$$
$$\text{d) } f_M(x) \equiv 1 \ \forall x \ge M \ (M = const > 0).$$

The conditions (7) are satisfied, for example, by means of the function [27]:

$$f_M(x) = \begin{cases} 1 - \exp\dfrac{Mx}{x-M}, & 0 \le x < M; \\ 1, & x \ge M. \end{cases} \quad (7)$$

Obviously, $f_M(x) \in C^\infty[0,\infty)$.

Let us denote $\omega_M(x,y) = f_M[\omega(x,y)]$.

It is easy to verify that the function $\omega_M(x,y)$ satisfies the conditions 1) – 3) of (5). Besides, $\omega_M(x,y)\equiv 1$, if $\omega(x,y)\ge M$.

Note that this condition means: if the function $\omega(x,y)$ is increasing monotonically while removing from the contour $\partial\Omega$, then the function $\omega_M(x,y)$ is different from unity only in some finite annular region $\{0\le\omega(x,y)<M\}$, that is lying outside $\bar{\Omega}$ and is adjacent to the $\partial\Omega$.

The following theorem has been proved.

Theorem. For any choice of sufficiently smooth functions Φ_1 and Φ_2

$$\left(\frac{\Phi_1}{\sqrt{x^2+y^2}}\to 0 \text{ as } \sqrt{x^2+y^2}\to+\infty\right)$$

the function

$$\psi=\omega_M^2(\psi_0+\Phi_1)+\omega_M^2(1-\omega_M)\Phi_2,$$

exactly satisfies the boundary conditions (2) and the condition at infinity (3). Here $\psi_0=U_\infty y\left(1-\dfrac{R^2}{x^2+y^2}\right)$ is the solution of the ideal fluid flow past circular cylinder of radius R (the cylinder of radius R entirely lies inside the streamlined body), $\omega_M=f_M(\omega)$, $f_M(\omega)$ has the form of (7), and ω is function with the properties of (5).

Problem 1. For approximating the indefinite components Φ_1 and Φ_2 it is proposed to use the Galerkin method [10]. The functions Φ_1 and Φ_2 will be presented as follows:

$$\Phi_1\approx\Phi_1^{m_1}=\sum_{k=1}^{m_1}\alpha_k\cdot\varphi_k,$$
$$\Phi_2\approx\Phi_2^{m_2}=\sum_{j=1}^{m_2}\beta_j\cdot\tau_j, \tag{8}$$

where: $\{\varphi_k(\rho,\varphi)\}=\left\{\rho^{2-k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, k=3,4,...;\right.$ $\left.\rho^{-k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, k=1,2,...\right\}$ is the complete system of partial solutions of the equation $\Delta^2\psi=0$ relative to the exterior of the cylinder of finite radius,

$$\{\tau_j(\rho,\varphi)\}=\left\{\cos 2\varphi,\sin 2\varphi,\rho^{j+2}\begin{matrix}\cos j\varphi\\ \sin j\varphi\end{matrix},\right.\quad \left.\rho^{j}\begin{matrix}\cos j\varphi\\ \sin j\varphi\end{matrix}, j=1,2,...\right\}$$

is the complete system of partial solutions of the equation $\Delta^2\psi=0$ for the region $\{\omega(x,y)<M\}$. Here $\rho=\sqrt{x^2+y^2}$ and φ is defined by the relations:

$$\cos\varphi=\frac{x}{\sqrt{x^2+y^2}},\ \sin\varphi=\frac{y}{\sqrt{x^2+y^2}}.$$

Thus, the approximate solution of the problem (1) – (3) is sought in the following form:

$$\psi\approx\psi_N=\omega_M^2(\psi_0+\Phi_1^{m_1})+\omega_M^2(1-\omega_M)\Phi_2^{m_2}. \tag{9}$$

Let us determine the complete sequence of functions relatively to the whole plain:

$$\{f_i(\rho,\varphi)\}=\left\{\omega_M^2(\rho,\varphi)\rho^{2-k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, k=3,4,...;\right.$$
$$\omega_M^2(\rho,\varphi)\rho^{-k}\begin{matrix}\cos k\varphi\\ \sin k\varphi\end{matrix}, k=1,2,...;$$
$$\omega_M^2(\rho,\varphi)\begin{matrix}\cos 2\varphi\\ \sin 2\varphi\end{matrix},\omega_M^2(\rho,\varphi)\rho^{j+2}\begin{matrix}\cos j\varphi\\ \sin j\varphi\end{matrix},$$
$$\left.\omega_M^2(\rho,\varphi)\rho^{j}\begin{matrix}\cos j\varphi\\ \sin j\varphi\end{matrix}, j=1,2,...\right\}. \tag{10}$$

The values of the coefficients α_k $(k=1,2,...,m_1)$ and β_j $(j=1,2,...,m_2)$ in accordance with the Galerkin method will be found from the condition of the residual $R_N=\Delta^2\psi_N+\mathrm{Re}\cdot A(\Delta\psi_N)$ orthogonality to the first N $(N=m_1+m_2)$ elements of the sequence (10):

$$(R_N,f_i)=0,\ i=1,2,...,N. \tag{11}$$

Besides, by the properties of ω_M and coordinate functions, the integration in (11) can be done only over a finite region $\{0\le\omega(x,y)<M\}$ when calculating the scalar products.

Problem 2. For solving the task (4), (2), (3) we propose to use a nonlinear Galerkin method. Approximate solution of problem (4), (2), (3) will be sought in the form (9), where Φ_1 and Φ_2 have the form (8). The values of the coefficients α_k and β_j will be found from the condition of the residual

Q_N orthogonality to the elements $f_1, \ldots, f_N$ of the sequence (10):

$$(Q_N,\ f_i)=0,\ i=\overline{1,\ N},\ N=m_1+m_2, \quad (12)$$

where:

$$Q_N=\Delta^2\psi_N-\mathrm{Re}\left(\frac{\partial\psi_N}{\partial y}\frac{\partial\Delta\psi_N}{\partial x}-\frac{\partial\psi_N}{\partial x}\frac{\partial\Delta\psi_N}{\partial y}\right).$$

Besides, by the properties of ω_M and coordinate functions, the integration in (12) can be done only over a finite region $\{0\le\omega(x,y)<M\}$ when calculating the scalar products.

As a result, we obtain a system of nonlinear equations, each of which is a quadratic function with respect to α_k and β_j. The resulting system can be solved by Newton's method. As an initial approximation, the set of α_k and β_j, corresponding to the solution of the Oseen problem or, at high Reynolds numbers, to the solution obtained at lower Reynolds numbers, can be chosen.

RESULTS

The computational experiment has been conducted for the problem of flow past the cylindrical body $x^8+y^8=1$ at $U_\infty=1$ and Reynolds numbers $\mathrm{Re}=0,01;\ 5;\ 10;\ 15;\ 20$.

Fig. 1 – 5 shows the streamline contours of the obtained approximate solution for $M=10$, $m_1=48$, $m_2=35$.

Detailed pictures of the streamline contours and vector velocity fields are shown in Fig. 6 – 10.

As can be seen from the figures, at low Reynolds numbers the flow is symmetric, has no separation zone in the aft area of the body. As Reynolds number is increased, the flow character is changed: the secondary vortices occur behind the body, their size and intensity grows, what coincides with physical experiments.

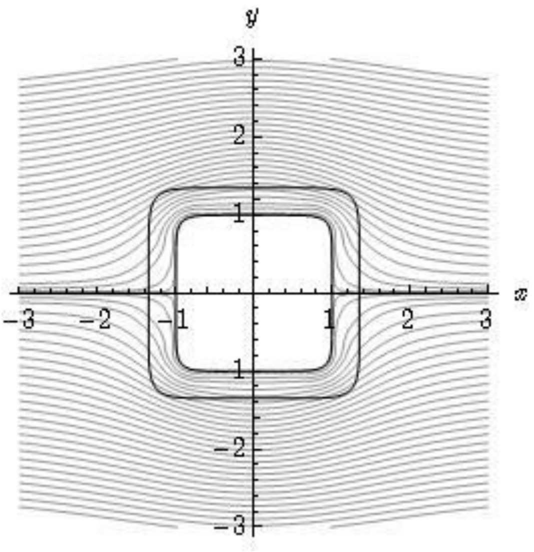

Fig. 1. The streamline contours at $\mathrm{Re}=0,01$

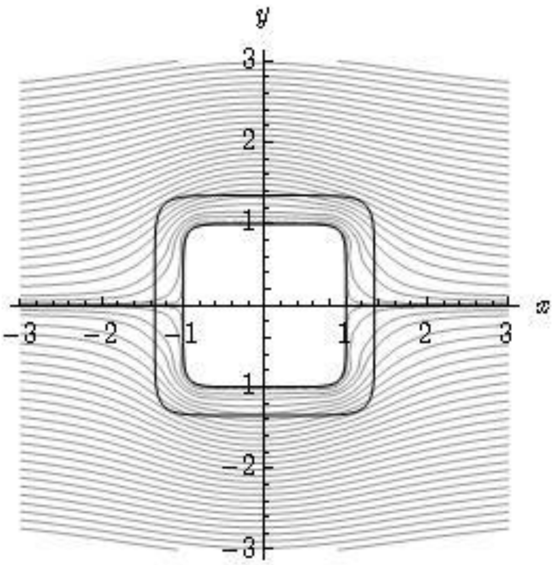

Fig. 2. The streamline contours at $\mathrm{Re}=5$

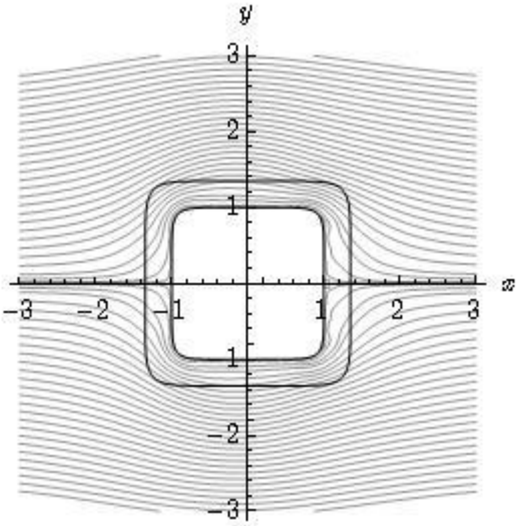

Fig. 3. The streamline contours at $\mathrm{Re}=10$

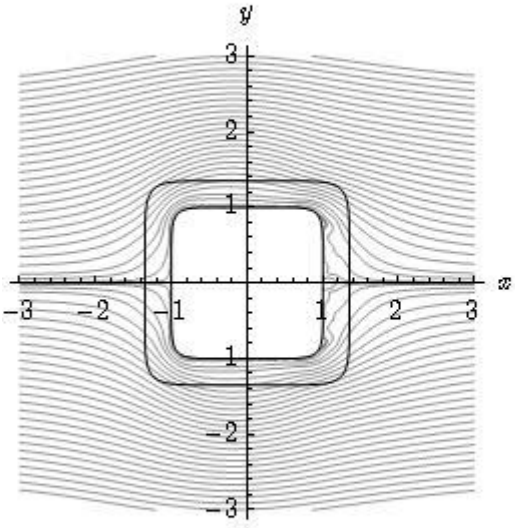

Fig. 4. The streamline contours at $\mathrm{Re}=15$

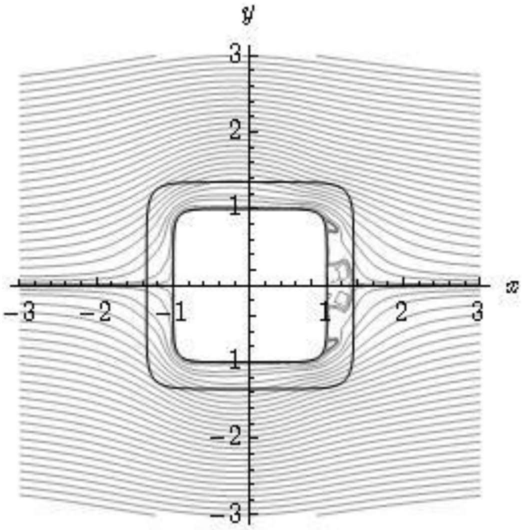

Fig. 5. The streamline contours at $\mathrm{Re}=20$

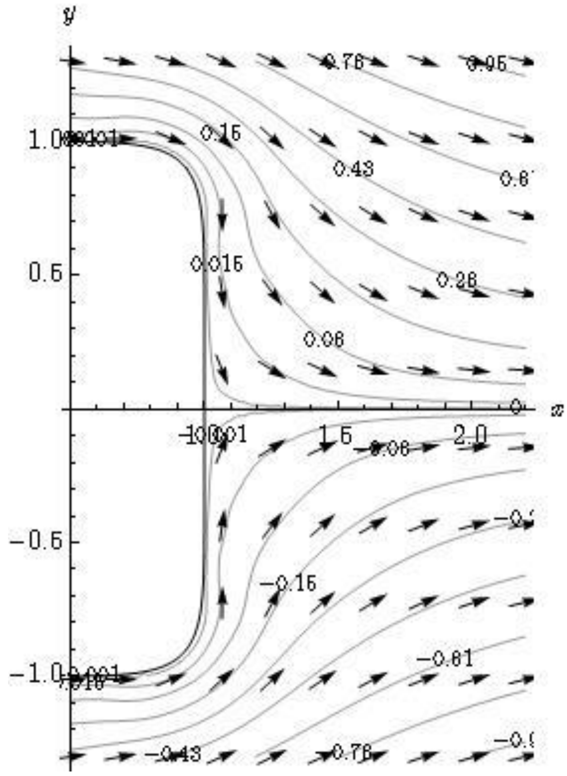

Fig. 6. Detailed pictures of the streamline contours and velocity vector fields at $\mathrm{Re}=0,01$

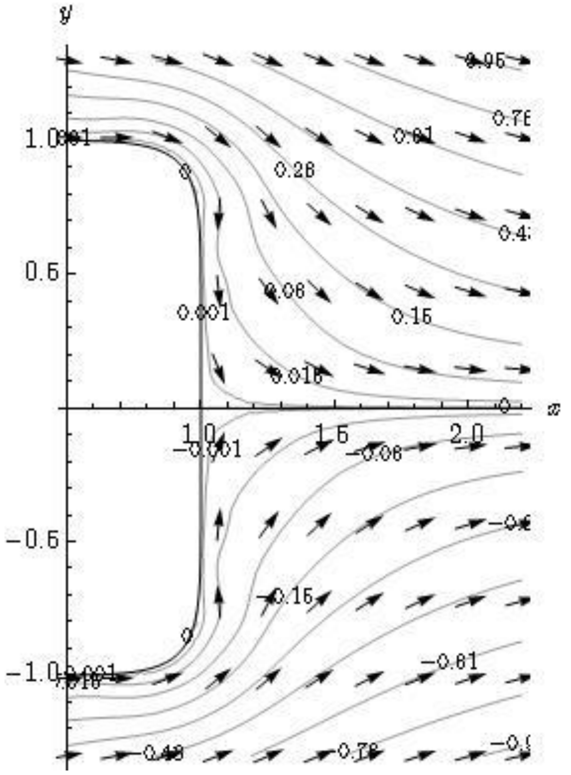

Fig. 7. Detailed pictures of the streamline contours and velocity vector fields at $\mathrm{Re} = 5$

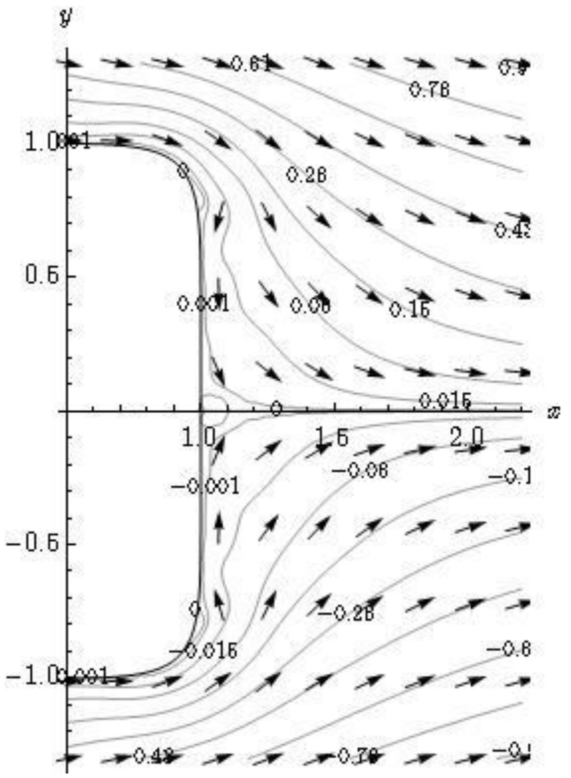

Fig. 8. Detailed pictures of the streamline contours and velocity vector fields at $\mathrm{Re} = 10$

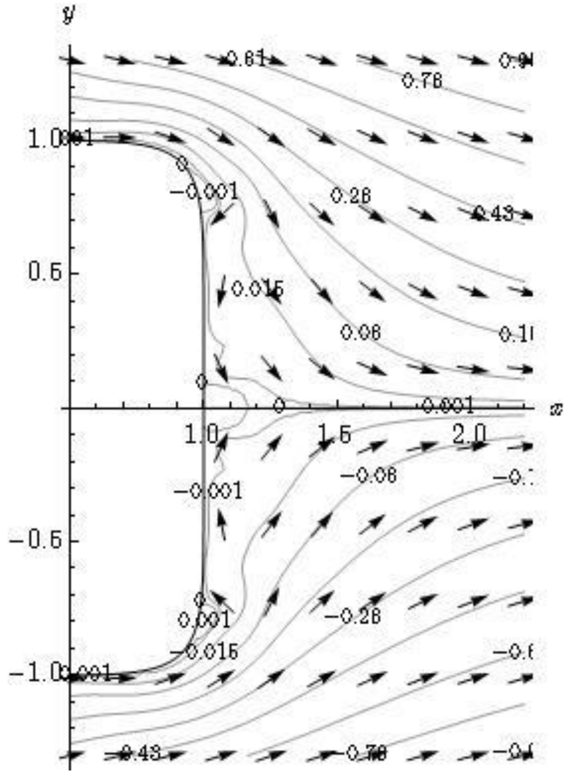

Fig. 9. Detailed pictures of the streamline contours and velocity vector fields at $\mathrm{Re} = 15$

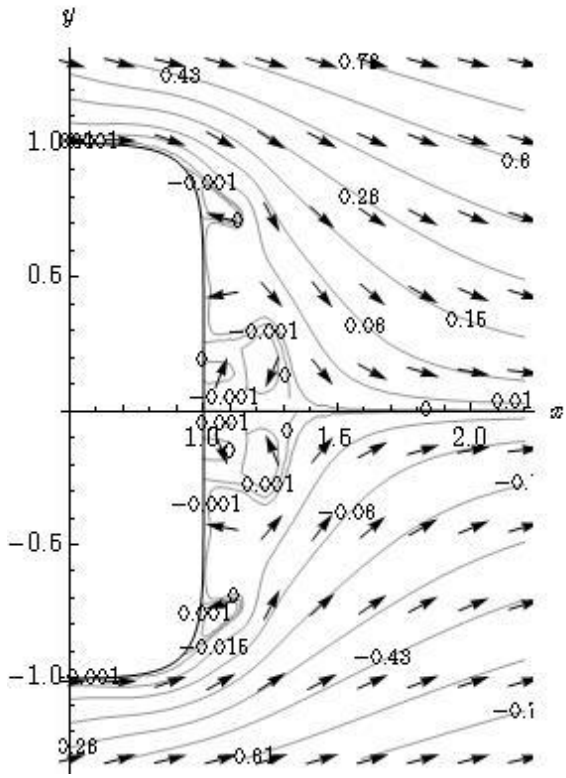

Fig. 10. Detailed pictures of the streamline contours and velocity vector fields at $\mathrm{Re} = 20$

CONCLUSIONS

The method for calculating the external flow of a viscous incompressible fluid, based on the joint use of the R-functions structural method and the Galerkin projection method, which differs from the known methods of universality (the algorithm does not change with changes in the geometry of the field) and the fact that the structure of the solution exactly takes into account the boundary conditions at the boundary of the body and the condition at infinity, has been proposed. For different Reynolds numbers the stationary problem of viscous incompressible fluid flow past the cylindrical body in a rectangular coordinate system has been solved numerically. The method developed allows to conduct the mathematical modeling of various biological, physical and mechanical flows.

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