

## STOCHASTIC OPTIMAL CONTROL OF A DESCRIPTOR SYSTEM

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**Abstract.** We study the optimal control problem for a descriptor system whose evolution is described by Ito's differential-algebraic equation. The quadratic cost functional is considered. The main constraint is that the characteristic matrix pencil corresponding to the equation is regular. We establish the conditions for the existence and uniqueness of the optimal control and the corresponding optimal state. The results are illustrated on an example of a descriptor system that describes transient states in a radio engineering filter with random perturbations in the form of white noise.

**Keywords:** stochastic differential algebraic equation, Wiener random process, quadratic cost functional, stochastic optimal control, radio engineering filter, transient state.

In the paper, we continue the studies started in [1, 2] and related to the theory of descriptor control systems and its applications. If a control system is subject to random perturbations, then the process of Brownian motion is the most comprehensible approach to model such perturbations [3, 4]. As a result, the model equation is Ito's stochastic differential equation. If a discrete model is subject to random perturbations, then stochastic difference equations are applied, for example, like for the conflict-controlled system from [5]. Presence of random perturbations in descriptor systems can be described by means of stochastic differential equations not solved with respect to the stochastic differential of the required random process [6–8]. In the present paper, we propose an analysis of the problem of stochastic optimal control of a descriptor system, which is shown on the example of minimizing energy functional of a radio engineering filter with regard for random perturbations.

Let us introduce the notation:  $\langle \cdot, \cdot \rangle$  is scalar product,  $\|\cdot\|$  is a norm,  $E$  is a unit matrix of appropriate dimension;  $B^{\text{tr}}$  is a transposed matrix or a transposed vector;  $\{\Omega, F, P\}$  is a complete probability space with nondecreasing family of sigma algebras  $\{F_t\}_{0 \leq t \leq T}$  ( $F_s \subseteq F_t \subseteq F, 0 \leq s \leq t \leq T$ );  $w(t)$  is a Wiener process with values in  $\mathbf{R}^{m^2}$ , outgoing from zero and coordinated with the family of sigma algebras  $\{F_t\}$ ;  $L_2(0, T; \mathbf{R}^n)$  is the space of vector functions with values in  $\mathbf{R}^n$ , summable with the squared norm on  $[0, T]$ ;  $W_2^k(0, T; \mathbf{R}^n)$  is the Sobolev space of vector functions from  $L_2(0, T; \mathbf{R}^n)$ , whose generalized derivatives of order up to  $k$  inclusively belong to  $L_2(0, T; \mathbf{R}^n)$ ;  $H_2(\Omega; \mathbf{R}^n) = H_2$  is the Hilbert space of  $n$ -dimensional random variables  $\xi = \xi(\omega)$  having a finite absolute moment of second order  $\mathbf{M}\|\xi\|^2 < \infty$ , with the scalar product  $\langle \xi, \eta \rangle_{H_2} = \mathbf{M}\langle \xi, \eta \rangle$  if  $F_0$  is a sigma subalgebra of the sigma algebra  $F$ , then  $H_2(\Omega; \mathbf{R}^n; F_0)$  is a subspace of space  $H_2$ , consisting of  $F_0$ -measurable random variables;  $L_2(0, T; \Omega; \mathbf{R}^n) = L_{2, \Omega, \mathbf{R}^n}$  is the Hilbert space of  $n$ -dimensional measurable random processes  $x(t, \omega), 0 \leq t \leq T, \omega \in \Omega$ , satisfying the condition  $\int_0^T \mathbf{M}\|x(t, \omega)\|^2 dt < \infty$ , with scalar product

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$$\langle x, y \rangle_{L_2, \Omega, \mathbf{R}^n} = \int_0^T \mathbf{M} \langle x(t), y(t) \rangle dt;$$

$L_2(0, T; \Omega; \mathbf{R}^n; F_t) = L_{2, \Omega, \mathbf{R}^n, F_t}$  is a subspace of space  $L_{2, \Omega, \mathbf{R}^n}$ , consisting of non-anticipating random processes with respect to the set of  $\sigma$ -algebras  $\{F_t\}_{0 \leq t \leq T}$ . We will assume that functions from  $W_2^1(0, T; \mathbf{R}^n)$  are continuous on  $[0, T]$  and will change them on a set of zero measure if necessary. The orthogonal projection  $\Pi y(t)$  of random process  $y(t) \in L_2(0, T; \Omega; \mathbf{R}^n)$  onto the subspace  $L_2(0, T; \Omega; \mathbf{R}^n; F_t)$  for almost all  $t \in [0, T]$  is the conditional expectation  $\mathbf{M}[y(t) | F_t]$  of random variable  $y(t)$  with respect to the sigma algebra  $F_t$ .

## 1. PROBLEM STATEMENT ON THE EXAMPLE OF A RADIO ENGINEERING FILTER

As an example of a stochastic descriptor control system, let us consider the electric circuit presented in Fig. 1.

Two voltage sources act at the circuit input:  $e(t) = e(t, \omega)$  is a given random process;  $u(t) = u(t, \omega)$  is a random process, which is a control. In parallel with the capacities  $C_1$  and  $C_2$ , conductivities  $g_1$  and  $g_2$  are included. In series with the inductance  $L$ , ohmic resistance  $r$  is included. As is common in radio engineering systems, random perturbations are taken into account by means of white noise, i.e., by the generalized derivative  $w'(t)$  of the Wiener process  $w(t)$  [9, 10]. We will model this situation according to the following procedure: white noise  $w'(t)$  appears additively in the differential relations between currents and voltages:

$$LI'_L = U_L + \tau_L w'(t), \quad C_j U'_{C_j} = I_{C_j} + \tau_{C_j} w'(t), \quad j=1, 2. \quad (1)$$

Taking into account that white noise is a generalized derivative of the random process and does not exist as a normal function, we rearrange relations (1) in the form of stochastic differentials as is common in the theory of stochastic differential equations [11]:

$$LdI_L(t) = U_L(t)dt + \tau_L dw(t), \quad C_j dU_{C_j}(t) = I_{C_j}(t)dt + \tau_{C_j} dw(t), \quad j=1, 2. \quad (2)$$

According to the Kirchhoff and Ohm laws for currents and voltages, we get

$$\begin{aligned} I_{C_1} + I_{g_1} = I_{C_2} + I_{g_2} + I_L, \quad U_{C_1} + U_L + U_r = e, \quad U_{C_1} + U_{C_2} + u = e, \\ U_r = rI_L, \quad I_{g_j} = g_j U_{C_j}, \quad j=1, 2. \end{aligned} \quad (3)$$

In relations (1)–(3), we suppose that  $L$ ,  $r$ ,  $C_j$ , and  $g_j$  are positive constants and  $\tau_L$  and  $\tau_{C_j}$  are weight factors that determine the values of influence of white noise on the respective inertial elements. The state of the electric circuit is characterized by the three-dimensional random process  $x(t) = x(t, \omega)$ , which consists of “energy” components corresponding to the inertial elements:

$$x(t) = x(t, \omega) = (I_L(t) \quad U_{C_1}(t) \quad U_{C_2}(t))^T. \quad (4)$$

Performing obvious transformations over relations (2)–(4), we obtain the system of three equations with respect to “energy” components (4) on the time interval  $[0, T]$

$$\begin{aligned} LdI_L(t) + U_{C_1}(t)dt + rI_L(t)dt = e(t)dt + \tau_L dw(t), \\ C_1 dU_{C_1}(t) - C_2 dU_{C_2}(t) - I_L(t)dt + g_1 U_{C_1}(t)dt - g_2 U_{C_2}(t)dt = (\tau_{C_1} - \tau_{C_2})dw(t), \\ U_{C_1}(t) + U_{C_2}(t) = e(t) - u(t). \end{aligned} \quad (5)$$

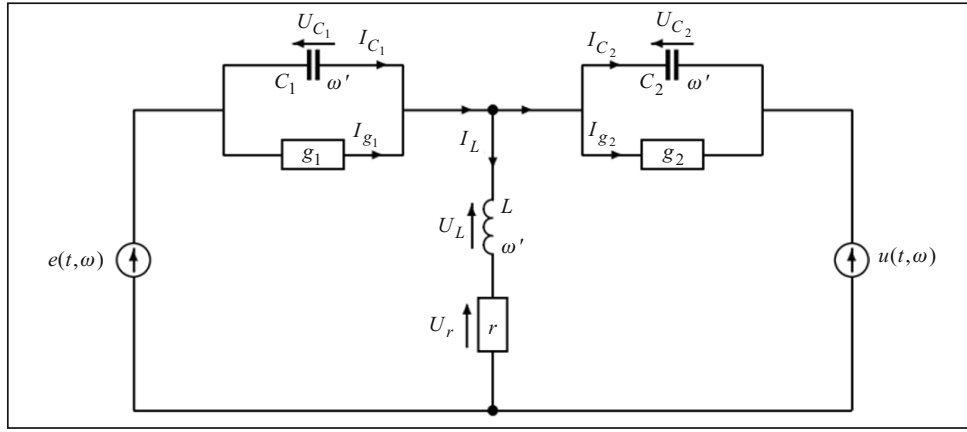


Fig. 1. Scheme of the radio engineering filter.

The first two equations are stochastic differential and the third one is algebraic. At the initial instant of time  $t=0$ , the values are given

$$LI_L(0, \omega) = \xi_1(\omega), \quad C_1 U_{C_1}(0, \omega) - C_2 U_{C_2}(0, \omega) = \xi_2(\omega), \quad (6)$$

where  $\xi_1$  and  $\xi_2$  are random variables. The problem of stochastic optimal control of system (5), (6) is to choose input (control) voltage  $u(t)$  that implements the minimum of the functional of energy of inertial elements and control

$$J(u) = \mathbf{M} \left\{ \int_0^T [LI_L^2(t) + C_1 U_{C_1}^2(t) + C_2 U_{C_2}^2(t) + u^2(t)] dt \right\}. \quad (7)$$

In the notation

$$A = \begin{pmatrix} L & 0 & 0 \\ 0 & C_1 & -C_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} r & 1 & 0 \\ -1 & g_1 & -g_2 \\ 0 & 1 & 1 \end{pmatrix},$$

$$f(t) = \begin{pmatrix} e(t) \\ 0 \\ e(t) \end{pmatrix}, \quad K = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} \tau_L \\ \tau_{C_1} & -\tau_{C_2} \\ 0 \end{pmatrix} \quad (8)$$

we rearrange the initial problem (5), (6) in vector form with respect to the state  $x(t)$  (4):

$$d[Ax(t)] + Bx(t)dt = f(t)dt + Ku(t)dt + \sigma dw(t), \quad 0 \leq t \leq T, \quad (9)$$

$$Ax(0, \omega) = \xi(\omega). \quad (10)$$

Unlike the respective radio engineering model from [10], the stochastic model (9), (10) is descriptor due to degeneracy of the matrix  $A$ .

Description of the mathematical models of circuits with segments of long lines and concentrated elements leads to descriptor systems with delay [12, 13]. Taking into account random action on elements of these circuits is modeled by stochastic descriptor systems with delay [7].

## 2. STATEMENT OF THE PROBLEM OF STOCHASTIC OPTIMAL CONTROL OF A DESCRIPTOR SYSTEM

Let us consider the problem of stochastic optimal control of system (9), (10) in a more general situation, where  $A$  and  $B$  are real matrices of dimensions  $n \times n$  and  $K$  and  $\sigma$  are real matrices of dimensions  $n \times m_1$  and  $n \times m_2$ , respectively,

$f(t)$  is an  $n$ -dimensional random process,  $u(t)$  is an  $m_1$ -dimensional random process (control),  $w(t)$  is an  $m_2$ -dimensional Wiener process, and  $\xi$  is an  $n$ -dimensional random variable. To estimate the quality of control of system (9), (10), we define the functional

$$J(u) = \mathbf{M} \left\{ \int_0^T [\langle Rx(t), x(t) \rangle + \langle Nu(t), u(t) \rangle] dt \right\}, \quad (11)$$

where  $R$  and  $N$  are real  $n \times n$  and  $m_1 \times m_1$  matrices.

Let us associate feasible control  $u(t) \in U = L_2(0, T; \Omega; \mathbf{R}^{m_1}; F_t)$  with system's state  $x(t) = x(t; u)$  or solution of the initial problem (9), (10). Let  $f(t, \omega) \in L_2(0, T; \Omega; \mathbf{R}^n; F_t)$ ,  $\xi \in H_2(\Omega; \mathbf{R}^n; F_0)$ . Random process  $x(t) \in L_2(0, T; \Omega; \mathbf{R}^n; F_t)$  is a solution of the initial problem (9), (10) if the equality holds

$$Ax(t) - \xi + \int_0^t Bx(s) ds = \int_0^t [f(s) + Ku(s)] ds + \sigma w(t) \quad \text{a.e. } t \in [0, T], \quad \omega \in \Omega. \quad (12)$$

Uniqueness of solution of stochastic differential equation is understood to within stochastic equivalence [11].

The problem of stochastic optimal control of system (9), (10) is to find the minimum

$$\min_{u \in U} J(u) \quad (13)$$

of the cost functional  $J(u)$  (11) on system's states  $x(t) = x(t; u)$ . We will call the control  $u_* \in U$  on which the minimum (13) is attained, i.e.,  $J(u_*) = \min_{u \in U} J(u)$ , the optimal control, and the state of system (9), (10) corresponding

to it  $x_*(t) = x(t; u_*)$  the optimal state or optimal solution.

In the analysis of control systems, important is representation of system's state that admits additive inclusion of the control unit, for example, in [14] or (for systems of a more general form) in [15]. Let us describe solutions  $x(t) = x(t; u)$  of the initial problem (9), (10) under various feasible controls  $u(t) \in U$ . The characteristic pencil of matrices or operators renders a significant influence on the dynamics of the descriptor system [16]. For the system (9), (10), such pencil is  $\lambda A + B$ . Most often it is supposed that the system is regular, i.e., the characteristic pencil is regular (pencil's determinant as a function of  $\lambda$  does not vanish identically). In what follows, we will assume that the descriptor system (9), (10) is regular. On the basis of the studies started in [1, 2], the regularity assumption allows introducing the following matrices:

$$I = \frac{1}{2\pi i} \oint_{\gamma} (\lambda A + B)^{-1} d\lambda, \quad G = AI(A - B) + B, \\ D = -AIBG^{-1}, \quad H = (E - AI)AG^{-1}, \quad (14)$$

where  $\gamma$  is the circuit that limits all the eigenvalues of the characteristic pencil. Note that the matrix  $I$  is real, matrices  $AI$  and  $IA$  are projective, and matrix  $H$  is nilpotent with the nilpotency index  $\nu$ .

In the theorem below, we will establish the conditions whereby the regular problem (9), (10) is uniquely solvable, and will also present an explicit formula to describe the states of the descriptor system or solutions of the given problem.

**THEOREM 1.** Let the following assumptions be satisfied for the regular system (9), (10):  $f(t, \omega) \in L_2(0, T; \Omega; \mathbf{R}^n; F_t)$ ;  $\xi \in H_2(\Omega; \mathbf{R}^n; F_0)$ ; if the nilpotency index  $\nu$  of matrix  $H$  is greater than 1, then deterministic are vector  $(E - AI)\xi$  and vector function  $Hf(t, \omega) = h(t) \in W_2^j(0, T; \mathbf{R}^n)$ ,  $j = 1, \dots, \nu - 1$ ; the following constraints hold:

$$HK = 0, \quad AI\sigma = \sigma, \quad (E - AI)\xi = \sum_{j=0}^{\nu-1} (-1)^j \frac{d^j}{dt^j} [H^j h(0)]. \quad (15)$$

Then to within stochastic equivalence, for any feasible control  $u(t) \in U$  it is possible to determine the unique state of system (9), (10), which admits representation in the form of the formula of variation of constants

$$\begin{aligned}
x(t) = x(t; u) &= \varphi(t) + G^{-1} \left\{ \int_0^t e^{D(t-s)} AIKu(s) ds + [E - AI]Ku(t) \right\}, \\
\varphi(t) &= G^{-1} \left\{ G^{-1} e^{Dt} AI\xi + \int_0^t e^{D(t-s)} AI f(s) ds \right\} \\
&+ G^{-1} \left\{ \sum_{j=0}^{\nu-1} (-1)^j \frac{d^j}{dt^j} [H^j (E - AI)f(t)] + \int_0^t e^{D(t-s)} \sigma dw(s) \right\}. \tag{16}
\end{aligned}$$

The stochastic integral  $\int_0^t e^{D(t-s)} \sigma dw(s)$  in formula (16) is understood in the sense of the Ito integral with respect to the vector Wiener process.

**Proof.** In the special case  $\nu = 1$ , the statement of Theorem 1 can be obtained from the respective results of [6] on the assumption that all the spaces are finite-dimensional. Let us consider the general situation  $\nu \geq 1$  and describe the scheme of the proof of Theorem 1. Since matrices  $AI$  and  $IA$  are projective, solution  $x(t)$  of problem (9), (10) admits representation in the form of the sum of two components  $x(t) = IAx(t) + (E - IA)x(t)$ , and Eq. (12) is equivalent to the system of two equations

$$GIAx(t) - AI\xi = \int_0^t DGIx(s) ds + \int_0^t AI[f(s) + Ku(s)] ds + \sigma w(t), \tag{17}$$

$$HG(E - IA)x(t) - (E - AI)\xi + \int_0^t G(E - IA)x(s) ds = \int_0^t (E - AI)[f(s) + Ku(s)] ds. \tag{18}$$

The component  $IAx(t)$  is a solution of Eq. (17) if and only if random process  $y(t) = GIAx(t)$  is a solution of the following initial problem for the linear stochastic differential equation:

$$dy(t) = Dy(t)dt + AI[f(t) + Ku(t)]dt + \sigma dw(t), \quad 0 \leq t \leq T, \quad y(0) = AI\xi.$$

This initial problem has a unique solution, which is strong according to the terminology from [11], and this solution is defined by the formula

$$y(t) = e^{Dt} AI\xi + \int_0^t e^{D(t-s)} AI[f(s) + Ku(s)] ds + \int_0^t e^{D(t-s)} \sigma dw(s). \tag{19}$$

By means of Eq. (18) and considering that  $\nu$  is the nilpotency index of matrix  $H$  and the first equality in (16) is true, we can uniquely find the component  $(E - IA)x(t)$  as

$$(E - IA)x(t) = G^{-1} \left[ \sum_{j=0}^{\nu-1} (-1)^j \frac{d^j}{dt^j} [H^j (E - AI)f(t)] + [E - AI]Ku(t) \right]. \tag{20}$$

From here, we obtain the necessary constraint in (15) for the random variable  $\xi$  (10).

Taking into account (19) and (20), we conclude that the initial problem (9), (10) has a unique solution  $x(t) = G^{-1}[y(t) + (E - IA)x(t)]$ , which can be presented in the form (16).

The theorem is proved.

The formula for states of system (16) does not envelop stochastic systems with impulse control; an interest to these systems is caused by the studies started in [2]. Taking into account the random and impulse components in [6] allows service of stochastic impulse systems if the nilpotency index  $\nu$  of matrix  $H$  of the descriptor system is equal to one or the system is explicit, i.e., the system equation is solved with respect to the state differential. This note specifies the method of the analysis of impulse systems from [17–19] with regard for random action in the form of additive white noise.

### 3. SOLUTION OF THE STOCHASTIC OPTIMAL CONTROL PROBLEM

In [6, 7], a method is proposed for the analysis of the stochastic optimal control problem with quadratic performance criterion for distributed systems not solved with respect to the stochastic differential  $dx(t)$ . Let us expand this method to the case of concentrated descriptor systems whose evolutionary equations are considered in finite-dimensional spaces. Unlike the studies [6, 7], we do not impose any constraints here on the inverse matrix  $(\lambda A + B)^{-1}$  and perform the analysis in real spaces.

**THEOREM 2.** Assume that for the regular system (9), (10) the assumptions of Theorem 1 are true. Let the matrices  $R$  and  $N$  in the definition of the cost functional (11) be nonnegative definite and moreover,  $N \geq \alpha E$ ,  $\alpha > 0$ . Then there exists a unique optimal control  $u_*(t) \in U$  on which the minimum (13) of the cost functional (11) is attained.

**Proof.** Let us present functional (11) as a quadratic form defined on the Hilbert space  $U = L_2(0, T; \Omega; \mathbf{R}^{m_1}; F_t)$ . Random process  $\varphi(t)$  in (16) is an element of space  $X = L_2(0, T; \Omega; \mathbf{R}^n; F_t)$ . Let us introduce a bounded linear operator

$$(\Psi z)(t) = G^{-1} \left\{ \int_0^t e^{D(t-s)} AIKv(s) ds + [E - AI]Kz(t) \right\}, \quad \Psi: U \rightarrow X,$$

and operator adjoint to it

$$(\Psi^* y)(t) = \Pi \int_t^T (G^{-1} e^{D(s-t)} AIK)^{\text{tr}} y(s) ds + [G^{-1}(E - AI)K]^{\text{tr}} y(t), \quad \Psi^*: X \rightarrow U.$$

Formula (16) for system's state becomes

$$x(t; u) = (\Psi u)(t) + \varphi(t).$$

Hence, the functional  $J(u)$  (11) admits the representation

$$J(u) = \langle R(\Psi u + \varphi), \Psi u + \varphi \rangle_X + \langle Nu, u \rangle_U. \quad (21)$$

By means of the self-adjoint bounded linear operator

$$\Phi = N + \Psi^* R \Psi: U \rightarrow U \quad (22)$$

we can rearrange expression (21) as

$$J(u) = \langle \Phi u, u \rangle_U + 2 \langle \Psi^* R \varphi, u \rangle_U + \langle R \varphi, \varphi \rangle_X.$$

Operator  $\Phi$  (22) has a bounded linear inverse operator  $\Phi^{-1}: U \rightarrow U$ , and for its norm the estimate  $\|\Phi^{-1}\| \leq \alpha^{-1}$  holds. We can immediately verify that for the control

$$u_* = -\Phi^{-1} \Psi^* R \varphi \quad (23)$$

for all  $u \in U$  the inequality holds

$$J(u) - J(u_*) = \langle \Phi(u - u_*), u - u_* \rangle_U \geq \alpha \|u - u_*\|_U^2.$$

From here it follows that control (23) is a unique optimal control of the problem (9)–(11), (13).

The theorem is proved.

In what follows, we assume that the conditions of Theorem 2, which ensure existence and uniqueness of the solution of the stochastic optimal control problem (9)–(11), (13), are satisfied. Being guided by the scheme of the analysis of distributed systems [20], we can find the conjugate state

$$p(t) = \int_t^T (G^{-1} e^{D(s-t)} AI)^{\text{tr}} R x(s) ds + [G^{-1}(E - AI)]^{\text{tr}} R x(t).$$

The conjugate state is a unique solution of the adjoint system

$$\frac{d}{dt}[(AIA)^{\text{tr}} p(t)] - B^{\text{tr}} p(t) = -Rx(t), \quad 0 \leq t \leq T, \quad (AIA)^{\text{tr}} p(T) = 0. \quad (24)$$

Equation (24) is a differential–algebraic equation with random right-hand side  $-Rx(t, \omega)$ . Its solutions are random processes  $p(t) \in L_2(0, T; \mathbf{R}^n)$   $P$  a.s. such that  $(AIA)^{\text{tr}} p(t) \in W_2^1(0, T; \mathbf{R}^n)$   $P$  a.s. and the equation is satisfied by  $P$  a.s. for a.a.  $t \in [0, T]$ .

Let us express optimal control in terms of the conjugate state. It can be easily seen that the relation

$$Nu + \Psi^* Rx(u) = 0$$

holds if and only if  $u$  is the optimal control and  $x$  is the optimal state of the problem (9)–(11), (13). From here, we obtain the representation for the optimal control

$$u_* = -N^{-1}K^{\text{tr}}\Pi p(t). \quad (25)$$

Therefore, for the optimal solution, Eq. (9) becomes

$$d[Ax(t)] + Bx(t)dt = [f(t) - KN^{-1}K^{\text{tr}}\Pi p(t)]dt + \sigma dw(t), \quad 0 \leq t \leq T. \quad (26)$$

The following theorem is true.

**THEOREM 3.** Assume that the assumptions of Theorem 2 are true. Then the boundary-value problem (26), (10), (24) has a unique solution  $x(t) = x_*(t) \in L_2(0, T; \Omega; \mathbf{R}^n; F_t)$ ,  $p(t) \in L_2(0, T; \Omega; \mathbf{R}^n)$ . Control (25) is the optimal control, and function  $x_*(t)$  is the corresponding optimal state of the problem (9)–(11), (13).

#### 4. APPLICATION TO THE ANALYSIS OF OPTIMAL TRANSIENT MODES OF A RADIO ENGINEERING FILTER

Let us apply the results obtained in Sec. 3 to the problem of stochastic optimal control of transient modes of a radio engineering filter, formulated in Sec. 1. Recall that the system (5), (6), which describes transient modes, admits representation (9), (10) in the notation (4), (8). The energy functional (7) can be presented in the form (11) with the matrix  $R = \text{diag}\{L, C_1, C_2\}$  and number  $N = 1$ . Let  $e(t) \in L_2(0, T; \Omega; \mathbf{R}; F_t)$ ,  $\xi_1, \xi_2 \in H_2(\Omega; \mathbf{R}; F_0)$ . The class of feasible controls (input voltages  $u(t)$ ) consists of random processes  $U = L_2(0, T; \Omega; \mathbf{R}; F_t)$ .

System (5), (6) is regular due to regularity of the characteristic determinant

$$\det(\lambda A + B) = (\lambda L + r)(\lambda C_{12} + g_{12}) + 1 \neq 0, \quad C_{12} = C_1 + C_2,$$

$$g_{12} = g_1 + g_2, \quad \mu C_{12} = C_2 g_1 - C_1 g_2.$$

Matrices in (14) become

$$AI = \begin{pmatrix} 1 & 0 & \frac{-C_2}{C_{12}} \\ 0 & 1 & -\mu \\ 0 & 0 & 0 \end{pmatrix}, \quad G^{-1} = \frac{1}{C_{12}} \begin{pmatrix} \frac{C_{12}}{L} & 0 & \frac{-C_2}{L} \\ 0 & 1 & C_2 - \mu \\ 0 & -1 & \mu + C_1 \end{pmatrix},$$

$$D = \frac{1}{LC_{12}} \begin{pmatrix} -rC_{12} & -L & rC_2 + \mu L \\ C_{12} & -g_{12}L & \mu L g_{12} - C_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Matrix  $H$  is zero one with the nilpotency index  $\nu=1$ . Let  $\xi = (\xi_1 \ \xi_2 \ 0)^{\text{tr}}$ . For the system (5), (6) with energy functional (7), conditions of Theorems 1–3 are satisfied. By virtue of Theorem 1, for any feasible control  $u(t) \in U$ , state  $x(t)$  (4) is defined uniquely in the explicit form

$$\begin{pmatrix} I_L(t) \\ U_{C_1}(t) \\ U_{C_2}(t) \end{pmatrix} = (\Psi u)(t) + \varphi(t), \quad (\Psi u)(t) = G^{-1} \int_0^t e^{D(t-s)} \begin{pmatrix} C_2 C_{12}^{-1} \\ \mu \\ 0 \end{pmatrix} u(s) ds - C_{12}^{-1} \begin{pmatrix} 0 \\ C_2 \\ C_1 \end{pmatrix} u(t),$$

$$\varphi(t) = G^{-1} \left[ e^{Dt} \begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \end{pmatrix} + \int_0^t e^{D(t-s)} \begin{pmatrix} C_1 C_{12}^{-1} \\ -\mu \\ 0 \end{pmatrix} e(s) ds + \int_0^t e^{D(t-s)} \begin{pmatrix} \tau_L \\ \tau_{C_1} - \tau_{C_2} \\ 0 \end{pmatrix} dw(s) \right] + C_{12}^{-1} \begin{pmatrix} 0 \\ C_2 \\ C_1 \end{pmatrix} e(t). \quad (27)$$

According to Theorem 2, there exists a unique optimal control  $u_*(t) \in U$  (23) on which minimum (13) of the energy functional (7) is attained. Then for the current  $I_L$  and voltages  $U_{C_1}$  and  $U_{C_2}$ , their optimal states  $I_{L^*}(t)$  and  $U_{C_1^*}(t)$ ,  $U_{C_2^*}(t)$  can be found from formulas (27) for  $u = u_*$  (25):

$$u = u_* = -\Phi^{-1} \Psi^* \text{diag} \{L, C_1, C_2\} \varphi, \quad (\Phi z)(t) = z(t) + (\Psi^* \text{diag} \{L, C_1, C_2\} \Psi z)(t),$$

$$(\Psi^* y)(t) = \mathbf{M} \left[ \int_t^T (G^{-1} e^{D(s-t)} A I K)^{\text{tr}} y(s) ds | F_t \right] - C_{12}^{-1} [C_2 y_2(t) + C_1 y_3(t)].$$

Optimal control and optimal state can be constructed, as in Theorem 3, by passage to the conjugate state

$$p(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} = \int_t^T (G^{-1} e^{D(s-t)} A I)^{\text{tr}} \begin{pmatrix} L I_L(s) \\ C_1 U_{C_1}(s) \\ C_2 U_{C_2}(s) \end{pmatrix} ds + \frac{C_1 C_2}{C_{12}} \begin{pmatrix} 0 \\ 0 \\ U_{C_1}(t) + U_{C_2}(t) \end{pmatrix}$$

and the adjoint system

$$\frac{d}{dt} [A^{\text{tr}} p(t)] - B^{\text{tr}} p(t) = - \begin{pmatrix} L I_L(t) \\ C_1 U_{C_1}(t) \\ C_2 U_{C_2}(t) \end{pmatrix}, \quad 0 \leq t \leq T, \quad p_1(T) = p_2(T) = 0. \quad (28)$$

Let us present Eq. (26) for the radio engineering system

$$\begin{aligned} L dI_L(t) + U_{C_1}(t) dt + r I_L(t) dt &= e(t) dt + \tau_L dw(t), \\ C_1 dU_{C_1}(t) - C_2 dU_{C_2}(t) - I_L(t) dt + g_1 U_{C_1}(t) dt - g_2 U_{C_2}(t) dt &= (\tau_{C_1} - \tau_{C_2}) dw(t), \\ U_{C_1}(t) + U_{C_2}(t) &= e(t) + \mathbf{M} [p_3(t) | F_t], \quad 0 \leq t \leq T. \end{aligned} \quad (29)$$

By virtue of Theorem 3, there exists a unique solution  $I_L(t) = I_{L^*}(t)$ ,  $U_{C_1}(t) = U_{C_1^*}(t)$ ,  $U_{C_2}(t) = U_{C_2^*}(t)$ ,  $p(t)$  of the boundary-value problem (29), (6), (28). By formula (25), we find the optimal control voltage  $u_*(t, \omega) = \mathbf{M} [p_3(t, \omega) | F_t]$ .

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