Mathematical Modeling of Radiation from a Periodical Multiple-Slot Superwide Antenna

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***Abstract*— Results of studying a model boundary value electromagnetic problem for a conical multi-element periodic structure in a rigorous formulation are presented. The problem solution method is based on using integral transformations and singular integral equations for the density of the induced surface current. The field distribution patterns in the wave zone are given in the case of exciting a cone with a longitudinal slot by pointed sources.**

***Keywords*— cone, slots, singular, integral equations, field, patters**

I. INTRODUCTION

Mathematical modeling of a process of interacting electromagnetic fields with objects of various configurations makes it possible to study their features and scattering properties in details. The results of studying the corresponding model problems can be effectively used for designing and creation of radio engineering devices and communication systems. Conical structures and their combanations are used in modern antenna technology, radar, communication systems as wideband and ultra-wideband antennas. [1, 2]. The operating range of the antenna can be extended by applying various surface inhomogeneities, special coatings and slots. Papers [3, 4] are devoted to investigating wave diffraction problems for solid perfectly conducting cones and the results of solving boundary electromagnetic problems for solid imperfectly conducting cones are given in [5, 6]. Studying the electromagnetic properties of perfectly conducting cones with azimuthal slots was carried out in [7, 8]. In this work studying the problem of exciting a semi-infinite multi-element periodic perfectly conducting cone by a point source is carried out for the first time. The goal of the work is to develop the singular integral equation method for solving electromagnetic multi-element cone boundary problems. The objectives of these studies are to obtain analytical and numerical solutions, as well as to study the effect of longitudinal slots on the main electromagnetic characteristics of the cone.

II. FORMULATION OF THE PROBLEM. METHOD OF SOLUTION

A semi-infinite perfectly conducting cone Σ with an opening angle 2γ and a periodic system of longitudinal slots cut along the generators of the system $N \in \boldsymbol{N}$ (Fig. 1) is in the field of a harmonic radial electric ($\chi = 1$) or a harmonic radial magnetic dipole ($\chi = 2$) that is located outside and at the axis of the structure under consideration. There are $M \in \boldsymbol{N}$ cone strips and M longitudinal slots of different angular sizes at the period of the cone $L = 2\pi / N$. The value of the slot width (angular size) is equal to the value of the dihedral angle formed by the planes drawn through the axis of the cone and the strip edges.

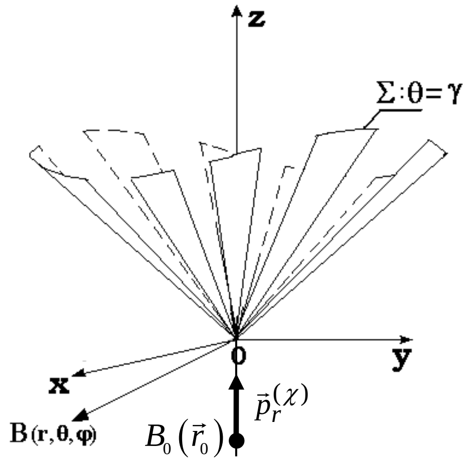


Fig. 1. Geometry of problem.

Let us denote $\vec{P}_r^{(\chi)}$ by the dipole,moment, d_p is the angular width of the slot with the number p ($p = 1 \div M$) at the period, $l_p^{(s)}$ is the strip with the number p at the period with the number $(s+1)$, $s = 0 \div (N-1)$. Let us introduce a spherical coordinate system (r, θ, φ) with the origin at the cone vertex. The conical surface Σ is defined as $\theta = \gamma$ in this coordinate system, and

$$l_p^{(s)} = \{\varphi : \rho \in (a_p + sL, b_p + sL)\},\ a_p, b_p \in R,$$

$$l^{(s)} = \bigcup_{p=1}^{M} l_p^{(s)}$$

is a total surface of all strips at the period with the number s,

$$l = \bigcup_{s=0}^{N-1} l^{(s)} = \bigcup_{s=0}^{N-1} \bigcup_{p=1}^{M} (a_p + sL, b_p + sL)$$

is a total surface of all strips of the conical surface Σ.

The problem is to find the total electromagnetic field

$$\vec{E}^{(\chi)} = \vec{E}_0^{(\chi)} + \vec{E}_1^{(\chi)},\ \vec{H}^{(\chi)} = \vec{H}_0^{(\chi)} + \vec{H}_1^{(\chi)}, \qquad (1)$$

where $\vec{E}_0, \vec{H}_0$ is a dipole field, and the unknown field $\vec{E}_1, \vec{H}_1$ is a cone presence result. The electromagnetic field $\vec{E}, \vec{H}$ satisfies Maxwell's equations everywhere outside the cone surface and the source, the boundary condition at the cone strips, the infinity condition , and the condition of limited energy. The boundary electromagnetic problem in this formulation has the unique solution.

To solve the problem we use the Debye potentials $v_0^{(\chi)}(\vec{r})$ $v_1^{(\chi)}(\vec{r})$ (the components of the field (1) are expressed via potentials) and reduce the original electromagnetic problem to solving the boundary value problem of mathematical physics for the potential $v_1^{(\chi)}(\vec{r})$.

We represent the potential $v_1^{(\chi)}(\vec{r})$ in the form

$$v_1^{(\chi)}(\vec{r}) = \frac{2}{\pi^2} \int_0^{+\infty} \tau \operatorname{sh} \pi\tau C_\tau^{(\chi)} V_\tau^{(\chi)}(\theta, \varphi) \frac{K_{i\tau}(qr)}{\sqrt{r}} dr, \quad (2)$$

where $K_{i\tau}(z)$ is the Mackdonald function, $q = -ik$, k – k is a wave number, $C_\tau^{(\chi)}$ are known coefficients,

$$V_\tau^{(\chi)} = \sum_{n=-\infty}^{+\infty} x_n^{(\chi)} \frac{P_{-1/2+i\tau}^{nN}(\pm\cos\theta)}{\dfrac{d^{\chi-1}}{d\gamma^{\chi-1}} P_{-1/2+i\tau}^{nN}(\pm\cos\gamma)} e^{inN\varphi}, \quad (3)$$

where $P_{-1/2+i\tau}^{nN}(\cos\theta)$ is the Legendre function, the upper signs in (3) correspond to the domain $0 < \theta < \gamma$, and the lower ones - to the domain $\gamma < \theta < \pi$, $x_n^{(\chi)}$ are unknown coefficients those are the solution of such functional equations

$$\sum_{n=-\infty}^{+\infty} x_n^{(\chi)} e^{inN\varphi} = 1, \qquad \varphi \in l^{(s)}, \qquad (4)$$

$$\sum_{n=-\infty}^{+\infty} [Nn]^{\rho(\chi)} \frac{|n|}{n} (1 - \varepsilon_n^{(\chi)}) x_n^{(\chi)} e^{inN\varphi} = 0,\ \varphi \in Cl^{(s)}, \qquad (5)$$

$$[Nn]^{\rho(\chi)} \frac{|n|}{n} (1 - \varepsilon_n^{(\chi)}) = \frac{(-1)^{(n+\nu)N+\chi-1} ch\pi\tau}{\pi(\sin\gamma)^{1-\rho(\chi)}} \frac{\Gamma(1/2 + i\tau + nN)}{\Gamma(1/2 + i\tau - nN)} \times$$

$$\times \frac{1}{\dfrac{d^{\chi-1}}{d\gamma_2^{\chi-1}} P_{-1/2+i\tau}^{(n+\nu)N}(\cos\gamma_2) \dfrac{d^{\chi-1}}{d\gamma_2^{\chi-1}} P_{-1/2+i\tau}^{(n+\nu)N}(-\cos\gamma_2)},$$

$$\rho(\chi) = (-1)^{\chi-1}$$

where $\Gamma(z)$ is the gamma-function. Equations (4), (5) are defined at different parts of the surface $\theta = \gamma$ and it gives significant difficulties for solving. We reduce (4), (5) to singular integral equations (SIE), each of which is defined either at strips or at slots. Let us introduce the function

$$F(\psi) = \sum_{n=-\infty}^{+\infty} |n| (1 - \varepsilon_n^{(1)}) x_n^{(1)} e^{in\psi},\ \psi \in [-\pi, \pi] \qquad (6)$$

$\psi = N\varphi$.

The coefficients $x_n^{(1)}$ are factors of the Fourier coefficients of the function $F(\psi)$. In the case of electric dipole excitation for finding $F(\psi)$ (6) we have the SIE of the form

$$\frac{1}{\pi} \int_{l^{(0)}} \ln\left|2\sin\frac{\psi - \xi}{2}\right| \cdot F(\xi) d\xi +$$

$$+ \frac{1}{\pi} \int_{l^{(0)}} R_{r\tau}(\psi - \xi) \cdot F(\xi) d\xi = -1,\ \psi \in l^{(0)}. \quad (7)$$

The resulting SIE (7) has a logarithmic singularity (the first term on the left side), while the second term has a regular kernel $R_{i\tau}(\psi - \xi)$ of the known form. The physical meaning of the function $F(\psi)$ is that it defines the radial component of the density of the surface current induced at the strips. In contrast to (4), (5) the SIE (7) is defined at the strips ($s = 0$) only.

To obtain the SIE in the case of magnetic radial dipole excitation we use the function:

$$\Phi(\psi_1) = \sum_{n=-\infty}^{\infty} \frac{|n|}{n} (1 - \varepsilon_n^{(2)}))(-1)^n x_n^{(2)} e^{in\psi_1},$$

$$\psi_1 = -\frac{|\varphi|}{\varphi} \pi + N\varphi,\ \psi_1 \in [-\pi, \pi]. \qquad (8)$$

Function $\Phi(\psi_1)$ (8) determines the radial component of the strip current density and is the solution of such an SIE

$$\frac{1}{\pi i} \int_{l^{(0)}} \frac{\Phi(\xi)}{\xi - \psi_1} d\xi + \frac{1}{\pi i} \int_{l^{(0)}} K(\xi - \psi_1) \Phi(\xi) d\xi = 1,\ \psi_1 \in S, \quad (9)$$

where $K(\xi - \psi_1)$ is a regular kernel.

III. ANALYTICAL SOLUTION FOR THE CASE OF A SEMITRANSPARENT CONE

Let us consider a cone, at the period of which there is one slot with an angular width d ($p = 1, d_p = d, \left|\vec{p}_r^{(\chi)}\right| = p_r^{(\chi)}$). The special case of the semitransparent cone is interesting for theory and practical applications. The semitransparent cone is determined by the existence of the limit ($\chi = 1$)

$$Q = \lim_{\substack{N \to \infty \\ d/L \to 1}} \left[-\frac{1}{N} \ln\cos\frac{\pi d}{2L} \right].$$

In this case the Deby’s potential (2) has the foolowing form:

$$v_1^{(1)} = \frac{\pi^2 p_r^{(\chi)}}{2r_0\sqrt{rr_0}} \int_{-i\infty}^{+i\infty} \frac{\mu \mathrm{T}_\mu\left(r,r_0\right)}{\Delta_\mu \cos\pi\mu} \left[P_{-1/2+\mu}(\cos\gamma)\right]^2 P_{-1/2+\mu}(-\cos\theta) d\mu \ ,$$

$$\gamma < \theta < \pi$$

$$\Delta_\mu = \pi P_{-1/2+\mu}(\cos\gamma) P_{-1/2+\mu}(-\cos\gamma) + 2Q\cos\pi\mu,$$

$$\mathrm{T}_\mu\left(r,r_0\right) = \begin{cases} J_\mu(kr) H_\mu^{(2)}\left(kr_0\right) , & r < r_0 \ , \\ H_\mu^{(2)}(kr) J_\mu\left(kr_0\right) , & r > r_0 \ ; \end{cases}$$

where $J_\mu(y)$ is a Bessel function, and $H_\mu^{(2)}\left(z\right)$ is a Hankel function of the second kind ($\mathrm{Im}\, k \le 0$). Electromagnetic field components satisfie the following averaged boundary conditions at the semitransparent cone surface:

$$E_r^+ = E_r^- \ , \quad -\frac{ik}{wQ\sin\gamma} E_r = \left(\frac{\partial^2}{\partial r^2} + k^2\right)\left(r\tilde{H}\varphi\right) \qquad (10)$$

$\tilde{H} = H^+ - H^-$, where f^+ и f^- mean the limiting values of the function f for $\theta = \gamma \pm 0$ respectively. The semitransparent cone is a model of a conical film that transmits and reflects electromagnetic field.s Averaged boundary conditions (10) are true at the conical film surface. The form of one of the electromagnetic field components in the domain $\gamma < \theta < \pi$: is given

$$E_{1\theta} = -E_{0\theta} - \frac{\pi^3 i p_1}{rr_0\sqrt{r_0}} \sum_{n=0}^{+\infty} \frac{\mu_n\left(\mu_n^2 - 1/4\right)\frac{d}{dr}\left(\sqrt{r}\mathrm{T}_{\mu_n}\left(r,r_0\right)\right)}{\cos\pi\mu_n \frac{d}{d\mu}\Delta_\mu\Big|_{\mu=\mu_n}} \times$$

$$\times\left[P_{-1/2+\mu_n}(\cos\gamma)\right]^2 P_{-1/2+\mu_n}^{-1}(-\cos\theta), \qquad \Delta_{\mu_n} = 0 \ .$$

The spectrum of the boundary eigenvalue problem for a semitransparent cone is a set of solution of the equation $\Delta_\mu = 0$. The smallest spectrum eigenvalue characterizes the field behavior near the cone vertex $(kr << 1)$. In case a) of a semitransparen cone, when Q is small, the roots of the equation with a small right hand side

$$\frac{\pi}{\cos\pi\mu} P_{-1/2+\mu}(\cos\gamma) P_{-1/2+\mu}(-\cos\gamma) = -2Q$$

are close to the solutions. $P_{-1/2+\mu}(\pm\cos\gamma) = 0$ [4].

$$\mu_q^\pm = \alpha_q^\pm -$$

$$-\frac{2Q\cos\pi\alpha_q^\pm}{\pi\frac{d}{d\mu}\left[P_{-1/2+\mu}(\cos\gamma)P_{-1/2+\mu}(-\cos\gamma)\right]\Big|_{\mu=\alpha_q^\pm}} +$$

$$+O\left(Q^2\right) ,$$

$$P_{-1/2+\alpha_q^+}(\cos\gamma) = 0 \ , \quad P_{-1/2+\alpha_q^-}(-\cos\gamma) = 0$$

Thus the eigenvalue spectrum in case a) is a perturbed eigenvalue spectrum of the Dirichlet boundar value problem for a isotropic cone [4]. In case b) of a semitransparent cone , $Q >> 1$, the eigenvalue spectrum is a set of μ_n :

$$\mu_n = 1/2 + n + \frac{1}{2Q}\left[P_n\left(\cos\gamma\right)\right]^2 + O\left(Q^{-2}\right) .$$

If the source is close to the cone vertex $\left(kr_0 << 1\right)$ one can use the following approximation for $E_{1\theta}$:

$$E_{1\theta}^* = \frac{ikp_r^{(1)}}{4Q}\left(\frac{kr_0}{2}\right)^{-3/2+\mu_0} \frac{\sin kr_0}{r_0} \cdot \frac{e^{-ikr}}{r} \mathrm{tg}\frac{\theta}{2} ,$$

$$r > r_0 \ , \quad 0 < \theta < \gamma .$$

This field approximation corresponds to a spherical TEM wave that propagates from the cone vertex. The field of this wave determines the behavior of the electromagnetic field near the cone vertex. There is no such wave in the field structure of an isotropic perfectly conducting cone, and its existence is due to the surface properties of a semitransparent cone. Along the cone axis this field is absent, and it reaches its maximum value at the cone surface. If the source is located at points $kr_0 = m\pi, m \in \boldsymbol{N}$, then the field of this wave is negligible small and its contribution to the total field is negligible too. From a practical point of view the excitation type is of interest when the source is located near the cone vertex $\left(kr_0 << 1\right)$. The expression for the radial component of the Umov–Poynting vector in this case has the form

$$S_r = \frac{\beta}{Q^2 r^2}\mathrm{tg}^2\frac{\theta}{2}, \quad 0 < \theta < \gamma, r > r_0 , \qquad (11)$$

where β is known coefficient. One can see from (11) that the energy flux maximum is at the conical surface.

IV. NUMERICAL RESULTS

If a numerical solution of SIE (7), (8) is obtained then the values of the Fourier coefficients $x_n^{(\chi)}$ for a cone with a longitudinal slot are found ($M = 1, N = 1$).

With an increase of the number n a decrease of $| x_n^{(\chi)} |$ is observed, which allows one to use a certain number of coefficients $x_n^{(\chi)}$ when calculating $V_\tau^{(\chi)}$ (3). The dependence $| x_0^{(1)} |$ on the angular dimensions of the conical structure is shown in Fig. 2.

Field distribution patterns in the wave zone ($\theta > 2\gamma$) depending on the slot width are shown in Fig. 3. When a cone with a longitudinal slot is excited by an electric dipole (Fig. 3.a) the slot expansion (to $d = 60^0$) does not significantly effect the field distribution.

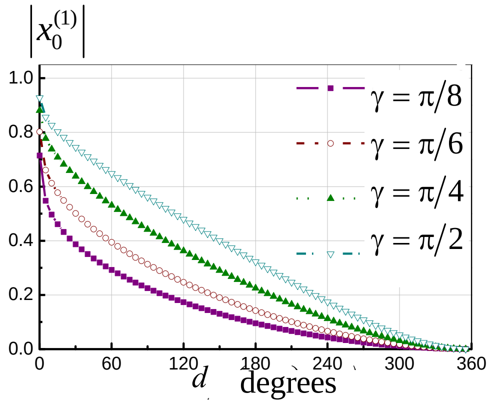
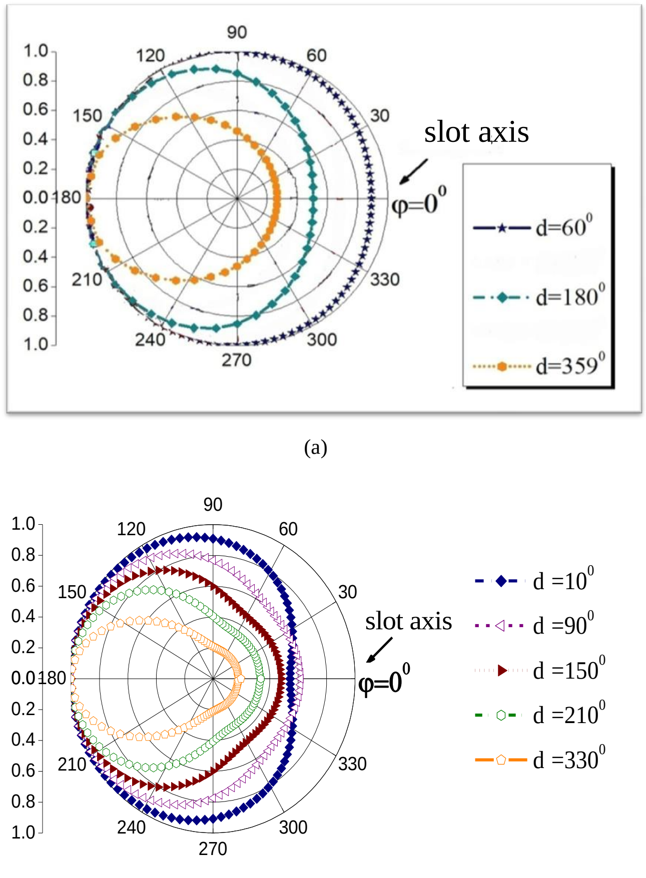


Fig. 2. Dependence of the value $\left|x_0^{(\chi)}\right|$ on the slot width d and the cone opening angle γ .



(a)

(b)

Fig. 3. Spatial distribution patterns of the field in the wave zone ($\theta > 2\gamma$, $\gamma = 22.5^0$, $\theta = 60^0$, $qr_0 = 1$) for a cone with a longitudinal slot: a) $\chi = 1$; b) $\chi = 2$.

Field distribution patterns of the slot cone that is excited by a magnetic dipole are given in Fig 3b. With the expansion of the gap, “glow” from the gap is observed, and the maximum falls at its middle. A further slot width increassng leads to a noticeable change in the shape of the pattern with the localization of the maximum in the middle of the conical strip, as a result of which the shape of the pattern approaches the shape of the pattern of a single extended scatterer, the axis of which is shifted relative to the axis by an angle.

V. CONCLUSIONS

In this paper, a model problem of point source excitation of a conical multi-element antenna is considered. The advantage of the proposed solution method is that the use of the mathematical apparatus allows us to reduce the problem posed to a singular integral equation defined at the strips of the entire structure period. However, in the particular case of narrow slots or narrow strips, one should use another method to obtain an asymptotic solution of the problem. In the case of a semitransparent cone an analytical solution of this problem is obtained, as well as averaged boundary conditions at the semitransparent conical surface. Field patterns in the wave zone depending on the angular width of the slot are given. The results obtained can be used in the design, creation of wideband and ultra-wideband antennas.

VI. REFERENCES

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