A Matrix Electrodynamics as an Analogue of the Heisenberg's Mechanics

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***Abstract*— A matrix approach to solving the electrodynamic problems is suggested. The specificity of one is treatment of an electrodynamic system (ES) as an oscillating system with a finite number of the degrees of freedom. The ES is considered as a set of spatially localized so-called partial oscillators (oscillets). Matrices of unit mutual pseudoenergies and unit mutual energies of the oscillators are evaluated. The eigenfrequencies and the eigenfunctions of the ES can be calculated basing on the lumped elements oscillating system matrix theory. A matrix second-order ordinary differential equation is solved for excited potentials of the ES instead of the D'Alembert equation. The main advantage of the matrix electrodynamics is substitution of the solving the partial derivative differential equations by the less computationally intensive linear algebra problems and the ordinary differential equation integration.**

I. Introduction

Fast expansion of digital technologies in communications causes some contradiction between practical requirements of the electronic engineering and respective theoretic abilities. E.g., many computational methods exist in the electrodynamics [1], nevertheless there are no perfect ones, which are fit in the enough degree for simulating, e.g., UWB electromagnetic potential excitation in various electrodynamic systems (ESs). The most popular FDTD/FDFD and FETD/FEFD methods are based on the lowest-order interpolation schemes, so they are rather "extensive". As a result, new numerical methods appear occasionally [2], and will appear in the future too. Possibly, this is profitable to seek ideas for new methods in other divisions of science.

E.g., two approaches to calculating the wave function Ψ are known in the quantum mechanics: the Schrödinger's wave mechanics and the Heisenberg's matrix mechanics [3]. The first is based on direct solving a partial differential equation (Klein-Gordon PDE or Schrödinger's PDE). The second works with a vector space produced as ensemble of all possible solutions of the same equations. If this space is predefined, the matrix mechanics can be reduced to the linear algebra problems and the ordinary differential equations (ODEs), which are less computationally intensive comparing with the direct numerical integration of PDEs.

The D'Alembert PDE for components of the electromagnetic potential is a particular case of the Klein-Gordon equation. Therefore, in the classic electrodynamics, two approaches to calculating spatio-temporal distributions of the potential also may exist. Let us call their as a "wave" and a "matrix" electrodynamics.

II. Mathematical Part

The well-known expansion of the electromagnetic potential in eigenfunctions of an ES is a special case of the matrix electrodynamics. However, eigenvalues and eigenfunctions are rather inappropriate as the base items for the practical calculations, because their finding is the most difficult part of an electrodynamic problem solving (except for the simplest geometries of the ESs). As an alternative to the eigenfunction basis, a set of spatially localized partial functions of the ES (synonyms: partial oscillators, oscillets) is proposed in [4].

Let us consider a closed or periodical simply-connected ES with perfectly conductive wall. The generic potential [5] of the ES $\mathfrak{A}(t,x,y,z)=\sqrt{\varepsilon_0\mu_0}\Phi$ or $\vec{A}$, where $\Phi(t,x,y,z)$ and $\vec{A}(t,x,y,z)$ are the scalar and the vector potentials in the Lorentz gauge, is a solution of the D'Alembert equation:

$$\varepsilon_0\mu_0\frac{\partial^2\mathfrak{A}}{\partial t^2}-\nabla^2\mathfrak{A}=\mu_0\mathfrak{J} \qquad (1)$$

within the ES volume V with homogeneous or periodical boundary conditions (BCs) on the wall. $\mathfrak{J}(t,x,y,z)=\rho\big/\sqrt{\varepsilon_0\mu_0}$ or $\vec{j}$ is the generic current density [5], where $\rho(t,x,y,z)$ and $\vec{j}(t,x,y,z)$ are the charge and the current densities.

Let us assume that the spectrum of the generic potential is finite in the wavenumber domain, i.e. $\left|\vec{k}\right|\le k_{max}<\infty$. In that case, the potential can be written as a finite series:

$$\mathfrak{A}(t,x,y,z)=\mathbf{u}_{\mathrm{p}}(t)\boldsymbol{\mathfrak{A}}_{\mathrm{p}}(x,y,z), \qquad (2)$$

which may be treated as some generalization of the Shannon-Whittaker series into spatial coordinates. $\boldsymbol{\mathfrak{A}}_{\mathrm{p}}$ is a vector (column) of N real partial functions of the ES; $\mathbf{u}_{\mathrm{p}}$ is a vector (column) of N instantaneous values of these functions; N is total number of the ES eigenvalues (squared wavenumbers) less or equal to k_{max}^2 including multiple ones.

The vector $\boldsymbol{\mathfrak{A}}_{\mathrm{p}}$ is a nontrivial solution of so-called intervalues problem for the ES:

$$\nabla^2\boldsymbol{\mathfrak{A}}_{\mathrm{p}}+\left[k_{\mathrm{p}}^2\right]\boldsymbol{\mathfrak{A}}_{\mathrm{p}}=0, \qquad (3)$$

which consists in finding non-zero functions $\mathfrak{A}_{pn}$ (n = 0, 1, …, N–1) satisfying (3) and homogeneous or periodical BCs

along with the matrix $\left[k_{\mathrm{p}}^2\right]$ spatially localizing all these functions. The spatial localization means that all components of $\mathfrak{A}_{pn}$ tend to zero fast enough as the distances from one's global peaks growth. The $N \times N$ intervalues matrix $\left[k_{\mathrm{p}}^2\right]$ contains N^2 squared interwavenumbers of the partial oscillators. The matrix eigenvalues repeat the lowest ES eigenvalues including multiples of ones.

First kind orthogonality for the partial functions [4] is:

$$\frac{\varepsilon_0}{2}\int_V dxdydz\mathfrak{A}_{\mathrm{p}}\mathfrak{A}_{\mathrm{p}}^T = \left[\tilde{W}_{\mathrm{p}}\right], \tag{4}$$

where T means a vector transpose; $\left[\tilde{W}_{\mathrm{p}}\right]$ is a $N \times N$ symmetrical matrix containing N^2 unit mutual pseudoenergies of the oscillators generic potentials.

Second kind orthogonality for the partial functions [4] is:

$$\begin{aligned}&\frac{1}{2\mu_0}\int_V dxdydz(-\nabla^2\mathfrak{A}_{\mathrm{p}})\mathfrak{A}_{\mathrm{p}}^T = \\ &= \frac{1}{2\mu_0}\int_V dxdydz\mathfrak{A}_{\mathrm{p}}(-\nabla^2\mathfrak{A}_{\mathrm{p}}^T) = \left[W_{\mathrm{p}}\right],\end{aligned} \tag{5}$$

where $\left[W_{\mathrm{p}}\right]$ is a $N \times N$ symmetrical matrix containing N^2 unit mutual energies of the oscillators generic potentials.

An equivalent of the Relay's formula for the partial oscillators is:

$$\left[k_{\mathrm{p}}^2\right] = \varepsilon_0\mu_0\left[W_{\mathrm{p}}\right]\left[\tilde{W}_{\mathrm{p}}\right]^{-1}, \tag{6}$$

where (–1) means a matrix inversion. The intervalues matrix can be diagonalized with appropriate $N \times N$ form-matrix of normal modes $[F]$ producing an eigenvalues matrix $\left[k_{\mathrm{e}}^2\right]$:

$$\left[k_{\mathrm{e}}^2\right] = [F]\left[k_{\mathrm{p}}^2\right][F]^{-1}. \tag{7}$$

In practice, special algorithms of matrix diagonalization exist, so there is no problem with $[F]$ fitting.

The ES eigenfunctions vector $\mathfrak{A}_{\mathrm{e}}(x, y, z)$ can be found as:

$$\mathfrak{A}_{\mathrm{e}} = [F]\mathfrak{A}_{\mathrm{p}}. \tag{8}$$

Relative values of functions $\mathfrak{A}_{em}$ ($m = 0, 1, \ldots, N–1$) are known as normalization of the oscillets (the matrix $[F]$ is presumed as orthogonal). Four possible types of the normalization are described in [4]. Those are: (i) amplitude normalization; (ii) energy normalization of first kind; (iii) energy normalization of second kind; and (iv) special normalization (e.g., truncated Gaussian). The normalization allows control the matrices $\left[\tilde{W}_{\mathrm{p}}\right]$ and $\left[W_{\mathrm{p}}\right]$ shapes and the spatial localization of the oscillets.

There are also several types of the 2D and 3D oscillets depending on their vector structures in the Euclidean space: (i) scalar; (ii) potential vectorial; (iii) solenoidal vectorial of one (in 2D space) or two (in 3D space) types; and (iv) transverse vectorial. All they are described in [4]. As an example, Figs. 1–3 show components of the scalar, the potential, and the solenidal 2D oscillets respectively ($c = 1/\sqrt{\varepsilon_0\mu_0}$).

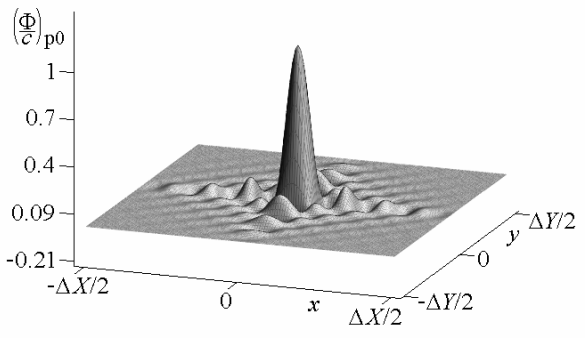

Fig. 1 Scalar 2D oscillet with the amplitude normalization

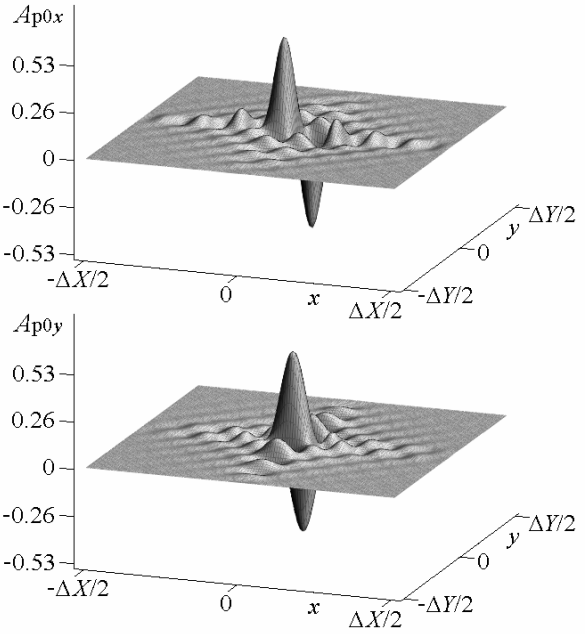

Fig. 2 Potential 2D oscillet with the amplitude normalization

A matrix ODE is used in the matrix electrodynamics for specification of excited potentials instead of PDE (1):

$$\left[\tilde{W}_{\mathrm{p}}\right]\frac{d^2\mathbf{u}_{\mathrm{p}}}{dt^2} + \left[W_{\mathrm{p}}\right]\mathbf{u}_{\mathrm{p}} = \mathbf{w}_{\mathrm{p}}, \tag{9}$$

where $\mathbf{w}_{\mathrm{p}}(t)$ is a vector (column) of partial energies of the generic currents in the ES volume:

$$\mathbf{w}_{\mathrm{p}} = \frac{1}{2}\int_V dxdydz\mathfrak{A}_{\mathrm{p}}\mathfrak{J}, \tag{10}$$

which are energies of the generic currents in unit generic potentials of respective partial oscillators.

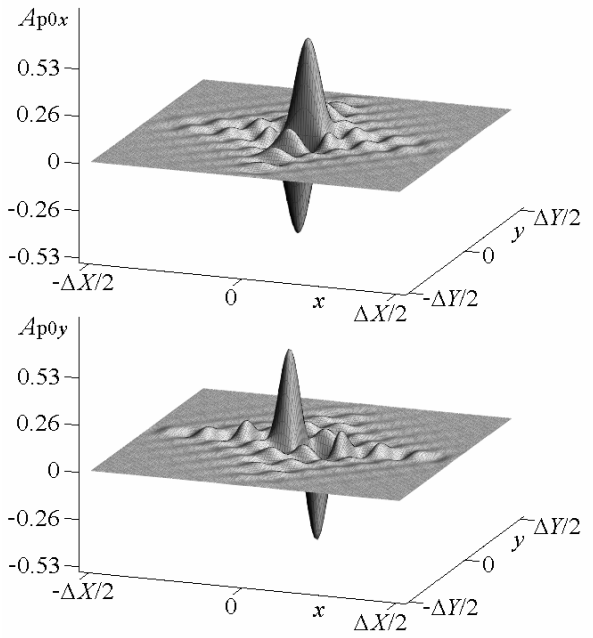

Fig. 3 Solenoidal 2D oscillet with the amplitude normalization

Formulas (2)–(10) are mathematical fundamentals of the matrix electrodynamics. From the physical point of view, the oscillet is a "cloud" of the electromagnetic potential oscillating as a single whole, i.e. in the same phase. Just so the matrix theory of a lumped element circuit with N degrees of freedom is applicable to the ES treating one as a 1D, 2D or 3D "lattice" of the coupled partial oscillators. Each partial function is an eigenfunction of a "partial" oscillating system obtaining from the original system with N degrees of freedom by freezing N–1 independent generalized coordinates (this explains the term "partial oscillator").

III. Algorithm of Calculations

Five fundamental features of the oscillets are given in [4]. Each solution of the intervalues problem has those features. However, an inverse supposition also can be made: any function having all those features might be one of solutions of the intervalues problem. Such assumption is of practical significance. If some of N partial functions of the ES are known a priori, the number of unknown ones in (3) reduces respectively. E.g., the oscillators allocated far from the ES boundaries are known a priori, as similar to the free-space oscillets. The oscillators lying close to a flat metal surface can be easy found too using "mirror reflections". Even in a "hard" case, when some oscillets lie near a non-flat surface, only a matrix initial-value problem (3) have to be solved for them instead of respective boundary-value problem, because this is supposed that all oscillets located close to a ES wall decrease practically to zero nearby the opposite wall.

Thus, obtaining spatial finitness of the partial functions, i.e. reducing their "significant" volumes to values much less than V, is an important problem. The amplitude normalization produces slowly decreasing oscillets, like $\sin(x)/x$ function (see Figs. 1–3). Two ways to confinement the partial oscillator in the space have been examined: (i) using the amplitude normalization with windowing by the Gaussian window in the space domain (Fig. 4); and (ii) using the truncated Gaussian normalization (Fig. 5). As a result, the second type of the restriction is recognized as more optimal as having more acceptable spectrum in the wavenumber domain [6].

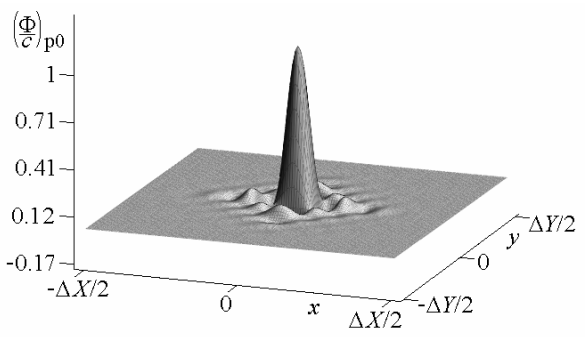

Fig. 4 Scalar 2D oscillet with the amplitude normalization windowed by the Gaussian window in the space domain

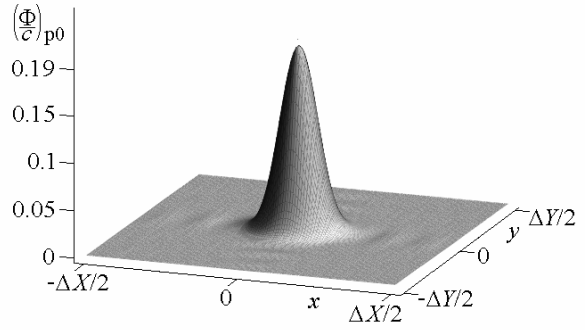

Fig. 5 Scalar 2D oscillet with the truncated Gaussian normalization

A general algorithm of calculations in the matrix electrodynamics is as follows. First, placing the oscillets within the ES volume is performed and adjustment of outermost ones to satisfy the BCs on the ES walls is carried out. Then, matrices of unit mutual pseudoenergies and unit mutual energies are evaluated. The intervalues matrix is calculated as (6). If an eigenvalue problem has to be solved, the eigenvalues matrix is obtained by $\left[k_{\mathrm{p}}^2\right]$ diagonalizing and vector of ES eigenfunctions is found as (8). Alternatively, matrix ODE (9) is solved in the time domain with calculating (10) at each time step to evaluate excited potentials.

IV. Numerical Results

An example of the eigenvalue problem is considered in this paper, as one of the most important electrodynamic problems, using the described technique. Two 1D oscillating systems both having 1024 units in the length and divided into 1024 intervals are simulated: (i) with the periodical BC, $N = 32$; and (ii) with the Neumann's BC, $N = 33$. A finite oscillet (Fig. 6) having 256 units in the length is used in both cases, which is

synthesized using the truncated Gaussian normalization. The amplitude spectrum of one in the basis of complex eigenfunctions of the closed-loop system is in Fig. 7. Two variants of the oscillet locations are examined: (i) regular with the distance of 32 units; and (ii) stochastic with a random deviation of each oscillet position no more than ±16 units from the "regular" location of one. The Neumann's BC for the boundary oscillets are simulated by superposition of "mirror reflections" of the outside fragments with further normalization of the peak values of the obtained sums.

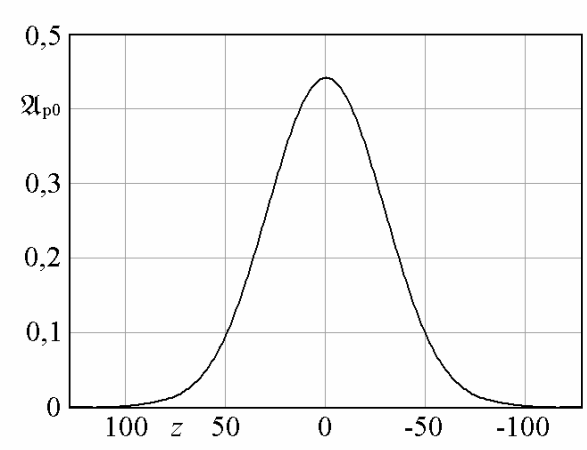

Fig. 6 1D oscillet with the truncated Gaussian normalization

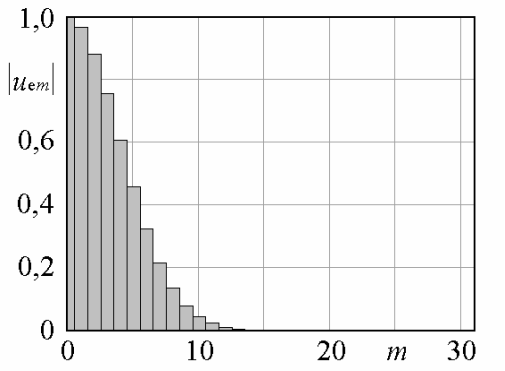

Fig. 7 Spectrum of the 1D oscillet with the truncated Gaussian normalization in basis of the oscillating system complex eigenfunctions

The results of simulations of the systems (i) and (ii) are given in Tables I and II respectively. Structures of both tables are similar. Column 1 is the eigenvalue numbers *m*. Column 2 shows the rigorous (analytically evaluated) eigenvalues. Columns 3 and 4 contain numerically calculated eigenvalues for the regular and the stochastic oscillet dispositions respectively with the described above technique.

As it can be seen from the tables, the regular location of the oscillets ensures almost perfect evaluation of the lowest roots. The stochastic position also gives good results for these values. The highest eigenvalues are evaluated less reliably. However, even in this case, the maximal relative error does not exceed 3…4 percent, i.e. is small enough from the point of view of prospective engineering applications.

TABLE I
EIGENVALUES OF THE 1D OSCILLATING SYSTEM WITH PERIODIC BC

1	2	3	4
0	+0.0000000	+0.0000000	+0.0000004
1	+0.0000376	+0.0000377	+0.0000377
2	+0.0000376	+0.0000377	+0.0000383
3	+0.0001506	+0.0001506	+0.0001507
4	+0.0001506	+0.0001506	+0.0001510
5	+0.0003388	+0.0003388	+0.0003389
6	+0.0003388	+0.0003388	+0.0003391
…	…	...	...
27	+0.0073793	+0.0073789	+0.0073946
28	+0.0073793	+0.0073789	+0.0074361
29	+0.0084711	+0.0085766	+0.0085294
30	+0.0084711	+0.0085766	+0.0087103
31	+0.0096383	+0.0096353	+0.0096803

TABLE II
EIGENVALUES OF THE 1D OSCILLATING SYSTEM WITH NEUMANN'S BC

1	2	3	4
0	+0.0000000	+0.0000000	+0.0000003
1	+0.0000094	+0.0000094	+0.0000096
2	+0.0000376	+0.0000377	+0.0000380
3	+0.0000847	+0.0000849	+0.0000852
4	+0.0001506	+0.0001509	+0.0001509
5	+0.0002353	+0.0002358	+0.0002358
6	+0.0003388	+0.0003395	+0.0003396
…	…	...	...
28	+0.0073793	+0.0073894	+0.0073958
29	+0.0079158	+0.0079417	+0.0079668
30	+0.0084711	+0.0085696	+0.0085492
31	+0.0090453	+0.0092720	+0.0093444
32	+0.0096383	+0.0096507	+0.0096766

V. CONCLUSIONS

The matrix electrodynamics is an equipollent alternative to the classical "wave" electrodynamics and shows considerable promise for the simulations of UWB electromagnetic potentials in various ESs.

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