TELECOMMUNICATIONS AND RADIO ENGINEERING

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CONTENTS

MICROWAVE ELECTRODYNAMICS

Wave Scattering on a Cone Surface with the Impedance-Type Boundary Conditions[*]

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INTRODUCTION

Placing on the bodies of distributed or concentrated impedance loads, through the use of which the surface of any shape would acquire the necessary scattering properties, is one of the efficient ways of controlling the scattering characteristics of various objects [1]. Covering of the bodies with radio absorbing substances is widely applied in measurement and control equipment, communication equipment, radars, medicine and other spheres of science and technology [2]. The scattering properties of the structure, upon the surface of which the absorbing layer is applied, is usually characterized by the reflection coefficient that can be determined from solving the model problem of electromagnetic waves scattering upon the considered structure. Availability of the impedance loads or the absorbing layer upon the surface of the body can be considered if, for instance, the averaged boundary conditions of the kind of the Shchukin-Leontovich impedance boundary conditions [3,4], are set upon the surface of this body. A number of papers, in which the scattering properties of the surfaces with impedance-type boundary conditions [5-7] have been studied by rigorous of approximation methods, are devoted to investigation of the model problems of electromagnetic waves scattering upon non-ideally conducting structures. The interest to investigation of the problems of scattering upon 3D and open-end structures has increased significantly during recently. Consideration of finite conductivity in such problems is essentially complicating their solution and requires to develop new or modifying the already existing solution methods.

The objective of the present paper is to develop the method based on attraction of the integral transformations technique and the method of semi-inversion for solving the problems of wave scattering upon an open-end cone, on the surface of which the impedance-type boundary conditions are set.

SETTING OF THE PROBLEM. PRESENTATION FOR SOLUTION.

Let us consider the scalar problem of waves scattering upon a semi-infinite thin circular cone Σ with slots periodically cut along the forming surface N (Fig. 1). The source of spherical monochromatic waves is placed into the point B_0. The particularity of the given cone structure is the fact that it possesses only angular geometrical parameters: 2γ is the aperture angle, d is the width of the slots, $l=2\pi/N$ is the period of the structure. Under the angular width of the slot and the period we understand the value of a dihedral angle formed by crossing of the planes directed through the axis of the cone and the edges of the adjacent cone strips (sectors). Considering that the cone is coinciding with one of the coordinate surfaces of the spherical system of coordinates, we introduce for the purpose of convenient solving this problem the spherical system of coordinates r, θ, φ with the beginning at the top of the cone, in which the surface of the cone is determined by the equation $\Sigma : \theta = \gamma$, it is required to find the scalar function (potential) $V^{(\chi)}$ determining the field in the presence of an open-end cone and the source (the parameter χ, $\chi = 1.2$ is corresponding to the cone with preset superficial properties depending upon χ).

The sought potential is satisfying the following:

1) Helmholtz equation $\Delta V^{(\chi)} - q^2 V^{(\chi)} = 0$ everywhere outside of the cone and the source $q = iak$, $a = \pm 1$; k is the wave number; dependence upon time is accepted in the form of $e^{ia\omega t}$;

2) boundary conditions on the strips of Σ

$$
\begin{cases}
L^{(\chi)}V^{(\chi)}\Big|_{\Sigma} = \\
= -R^{(\chi)}q^{\alpha(\chi)} \cdot \left\{ L^{(\chi+\alpha)}V^{(\chi)+} - L^{(\chi+\alpha)}V^{(\chi)-} \right\}, \\
L^{(\chi)}V^{(\chi)+} = L^{(\chi)}V^{(\chi)-};
\end{cases}
\tag{1}
$$

$$
L^{(1)}V = \left(\frac{\partial^2}{\partial r^2} - q^2 \right)(rV), \quad V^+ = V\big|_{\theta=\gamma+0},
$$

$$
L^{(2)}V = \frac{\partial V}{\partial \theta}, \quad V^- = V\big|_{\theta=\gamma+0}, \quad \alpha = \alpha(\chi) = (-1)^{\chi-1},
$$

where $R^{(z)}$ is the impedance parameter;
3) the saturable absorption principle;
4) energy finiteness condition.
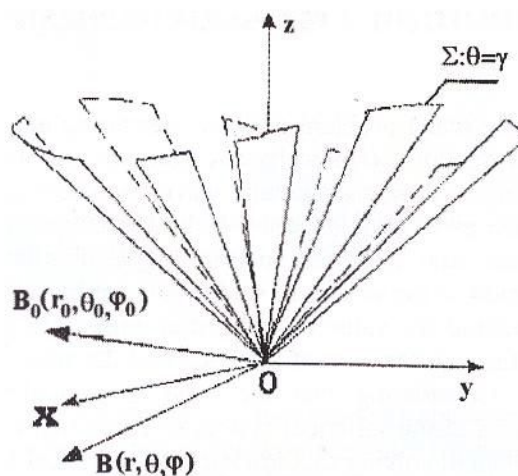
FIGURE 1.

Let us represent $V^{(z)}$ in the form of $V^{(z)} = V_0^{(z)} + V_1^{(z)}$, where $V_0^{(z)}$, $V_1^{(z)}$ is matching with the field of the source stipulated by the presence of a cone.

In order to solve the equation set we shall use pair of the Kontorovich–Lebedev integral transformations:

$$G(\tau) = \int\limits_0^{+\infty} g(r) \frac{K_{i\tau}(qr)}{\sqrt{r}} dr , \qquad (2)$$

$$g(r) = \frac{2}{\pi^2} \int\limits_0^{+\infty} \tau\, sh\pi\tau\, G(\tau) \frac{K_{i\tau}(qr)}{\sqrt{r}} d\tau , \qquad (3)$$

where $K_{i\tau}(qr)$ is the Macdonald function, and we put down the potential of the source like

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$$V_0^{(\chi)} = \frac{2}{\pi^2} \int\limits_0^{+\infty} \tau sh\pi\tau \hat{V}_0^{(\chi)} \frac{K_{i\tau}(qr)}{\sqrt{r}} d\tau,$$

$$\hat{V}_0^{(\chi)} = \sum\limits_{m=-\infty}^{+\infty} a_{m\tau}^{(\chi)} \cdot F(\theta,\theta_0,m,\tau) e^{im\varphi} \qquad (4)$$

$$F(\theta,\theta_0,m,\tau) = \begin{array}{l} P_{1/2+i\tau}^m(\cos\theta) P_{1/2+i\tau}^m(\cos\theta_0), \quad \theta < \theta_0, \\ P_{1/2+i\tau}^m(\cos\theta) P_{1/2+i\tau}^m(\cos\theta_0), \quad \theta_0 < \theta. \end{array}$$

In accordance with (2) to (4) we seek for the unknown potential $V_1^{(\chi)}$ in the form of Kontorovich–Lebedev integral:

$$V_1^{(\chi)} = \frac{2}{\pi^2} \int\limits_0^{+\infty} \tau sh(\pi\tau) \hat{V}_1^{(\chi)} \frac{K_{i\tau}(qr)}{\sqrt{r}} d\tau, \quad \hat{V}_1^{(\chi)} = \int\limits_0^{+\infty} \hat{V}_1^{(\chi)} \frac{K_{i\tau}(qr)}{\sqrt{r}} dr; \quad (5)$$

$$\hat{V}_I^{(\chi)} = -\sum\limits_{m=-\infty}^{+\infty} a_{m\tau}^{(\chi)} \hat{b}_{m\tau}^{(\chi),p} U_{m\tau}^{(\chi)}(\theta,\phi),$$

where the coefficient $\hat{b}_{m\tau}^{(\chi),p} = \dfrac{d^{\chi-1}}{d\gamma^{\chi-1}} F(\gamma,\theta_0,m,\tau) \cdot \eta\left((1-)^p (\theta_0 - \gamma)\right)$

contains the parameter of source location:

$$p = 1, \quad \theta_0 < \gamma; \quad \hat{b}_{m\tau}^{(\chi),1} = \frac{d^{\chi-1}}{d\gamma^{\chi-1}} P_{-1/2+i\tau}^m(-\cos\theta) P_{-1/2+i\tau}^m(\cos\theta_0),$$

$$p = 2, \quad \gamma < \theta_0; \quad \hat{b}_{m\tau}^{(\chi),2} = \frac{d^{\chi-1}}{d\gamma^{\chi-1}} P_{-1/2+i\tau}^m(\cos\theta) P_{-1/2+i\tau}^m(-\cos\theta_0),$$

where $P_{-1/2+i\tau}^m(\cos\theta)$ is the attached Lejandr function of the first kind, $\eta(y)$ is the Heaviside function;

$$U_{m\tau}^{(\chi)}(\theta,\varphi) = \sum\limits_{n=-\infty}^{+\infty} x_{m,n+m_0}^{(\chi)} \frac{P_{-1/2+i\tau}^{m+nN}(\pm\cos\theta)}{\dfrac{d^{j-1}}{d\gamma^{j-1}} P_{-1/2+i\tau}^{m+nN}(\pm\cos\gamma)} e^{i(m+nN)\varphi}, \quad (6)$$

where the sign "+" in (6) corresponds to the domain $0 < \theta < \gamma$, and "-" – to the domain $\gamma < \theta < \pi$, $v = m/N - m_0$, m_0 is the nearest integer number to m/N, $-1/2 \le v < 1/2$, $x_{m,n+m_0}$ are the unknown coefficients.

Thus, the initial boundary problem is reduced to finding of the Fourier coefficients $x_{m,n+m_0}$, the functional equations for which are obtained as the result of application to (5) of the boundary condition (1) as well as of the condition of continuity of the scattered field potential in the slots.

Consequently we assume that the impedance parameter $R^{(\chi)}$ depends upon the radial coordinate:

$$R^{(\chi)} = \frac{\varsigma^{(\chi)}}{(qr)^{\alpha(\chi)}}, \quad \alpha(\chi) = (-1)^{\chi-1}. \tag{7}$$

Introduction upon the formula (7) of a new impedance parameter $\varsigma^{(\chi)}$, which is independent upon the coordinates, provides for the possibility to separate the variables in the boundary condition (1) and to simplify the solution to the problem.

SET OF LINEAR EQUATIONS

Due to application of the boundary condition and the condition of continuity in the slots we proceed to the functional equations for the Fourier coefficients in the following form:

$$\sum_{m=-\infty}^{+\infty} \left\{ 1 + \varsigma^{(\chi)} \frac{1}{\left(\tau^2 + 1/4\right)^{\alpha(\chi)}} \frac{2\left[N(n+v)\right]^{\alpha(\chi)}}{\sin^{\alpha(\chi)}\gamma} \frac{|n|}{n}\left(1 - \varepsilon_n^{(\chi)}\right) \right\} \times \tag{8}$$

$$\times x_{m,n}^{(\chi)} e^{inN\varphi} = e^{im_0 N\phi} ribbons,$$

$$\sum_{m=-\infty}^{+\infty} \left[N(n+v)\right]^{\alpha(\chi)} \cdot \frac{|n|}{n}\left(1 - \varepsilon_n^{(\chi)}\right) x_{m,n}^{(\chi)} e^{inN\varphi} = 0, slots, \tag{9}$$

$$\left[N(n+v)\right]^{\alpha(\chi)} \cdot \frac{|n|}{n}\left(1-\varepsilon_n^{(\chi)}\right) =$$

$$= \frac{(-1)^{(n+v)N+\chi-1} ch(\pi\tau)}{\pi(\sin\gamma)^{1-\alpha(x)}} \cdot \frac{\Gamma\left(1/2+i\tau+(n+v)N\right)}{\Gamma\left(1/2+i\tau-(n+v)N\right)} \times$$

$$\times \frac{1}{\dfrac{d^{1-\chi}}{d\gamma^{1-\chi}} P_{-1/2+i\tau}^{(n+v)N}(-\cos\gamma) \dfrac{d^{1-\chi}}{d\gamma^{1-\chi}} P_{-1/2+i\tau}^{(n+v)N}(-\cos\gamma)},$$

where $\Gamma(z)$ is the gamma-function.

The system (8), (9) is a set of the first kind linear equations and possesses a number of drawbacks, e. g. instability. There exist different methods for solving such systems and the regularization method [5,8] is one of them. As the result of that, the set of the first kind is reduced to the system of linear equations of the second kind (SLAE-2), which is deprived from the drawbacks inherent to the set of the first kind. In the case of a boundary condition on the strips of the cone of the type

$$(\chi=2)\begin{cases} L^{(2)}V^{(2)}\Big|_{\Sigma} = \dfrac{-R^{(2)}}{q} \cdot \left(L^{(1)}V^{(2)+} - L^{(1)}V^{(2)-}\right) \\ L^{(\,)}V^{(1)+} = L^{(1)}V^{(1)-}, \end{cases}$$

we succeed in using the known procedure of regularization of the system (8), (9) and reduce the latter to SLAE-2 with respect to some unknown coefficients $x_{m,n}^{(2)}$. We rewrite (8), (9) for the case when $\chi=2$:

$$\sum_{n=-\infty}^{+\infty}\left\{1+\varsigma^{(2)}2\left(\tau^2+1/4\right)\sin\gamma\frac{1}{N(n+v)}\frac{|n|}{n}\left(1-\varepsilon_n^{(2)}\right)\right\}x_{m,n}^{(2)}e^{iNn\varphi} = \tag{10}$$

$$= e^{im_0 N\phi}, \qquad \pi d/l < |N\varphi| \leq \pi,$$

$$\sum_{n=-\infty}^{+\infty}\frac{1}{N(n+v)}\frac{|n|}{n}\left(1-\varepsilon_n^{(2)}\right)x_{m,n}^{(2)}e^{iNn\varphi} = 0, \quad |N\varphi| < \pi d/l. \tag{11}$$

We introduce into the consideration

$$y_n^{(2),m_0} = \left[1 + \varsigma^{(2)} 2\left(\tau^2 + 1/4\right) \sin\gamma \frac{1}{N(n+\nu)} \frac{|n|}{n} \left(1 - \varepsilon_n^{(2)}\right)\right] x_{m,n}^{(2)} - \delta_n^{m_0},$$

$$\delta_n^{m_0} = \begin{cases} 1, n = m_0 \\ 0, n \ne m_0 \end{cases}$$

and transform (10), (11) into the form, which is convenient for application of the regularization procedure:

$$\sum_{n=-\infty}^{+\infty} y_n^{(2),m_0} e^{in\psi} = 0, \qquad \pi d/l < |\psi| \le \pi, \tag{12}$$

$$\sum_{n=-\infty}^{+\infty} \frac{|n|}{n} y_n^{(2),m_0} e^{in\psi} = \sum_{n=-\infty}^{+\infty} f_n^{*(2),m_0} e^{in\psi}, \quad |\psi| < \pi d/l, \tag{13}$$

$$f_n^{*(2),m_0} = -\tilde{h}_{i\tau}^{(n+\nu)N} \delta_n^{m_0} + \tilde{f}_{i\tau}^{(n+\nu)N} y_n^{(2),m_0},$$

$$\tilde{h}_{i\tau}^{(n+\nu)N} = \frac{1}{1 + \varsigma^{(2)} 2(\tau^2 + 1/4)\sin\gamma \tilde{D}_{i\tau}^{(n+\upsilon)N}} \frac{|n|}{n}\left(1 - \varepsilon_n^{(2)}\right),$$

$$\tilde{f}_{i\tau}^{(n+\nu)N} = \frac{|n|}{n}\varepsilon_n^{(2)} + N(n+\nu)\frac{\varsigma^{(2)} 2(\tau^2 + 1/4)\sin\gamma(\tilde{D}_{i\tau}^{(n+\upsilon)N})^2}{1 + \varsigma^{(2)} 2(\tau^2 + 1/4)\sin\gamma \tilde{D}_{i\tau}^{(n+\upsilon)N}},$$

$$\psi = N\phi, \quad \tilde{D}_{i\tau}^{(n+\upsilon)N} = \frac{1}{N(n+\nu)} \frac{|n|}{n}\left(1 - \varepsilon_n^{(2)}\right).$$

Using the semi-inversion method we reduce the set of the first kind (12), (13) to SLAE-2:

$$\frac{1}{N} \frac{2P_{\nu-1}(-\upsilon)}{P_\nu(-\upsilon) + P_{\nu-1}(-\upsilon)} y_0^{(2),m_0} =$$

$$= -\tilde{h}_{i\tau}^{(m_0+\upsilon)N} \frac{\nu}{N} \tilde{V}^{m_0}(\upsilon) + \frac{\nu}{N} \sum_{p=-\infty}^{+\infty} \frac{|n|}{n} y_0^{(2),m_0} \tilde{f}_{i\tau}^{(p+\nu)N} \tilde{V}^p(\upsilon), \tag{14}$$

$$\tilde{y}_0^{(2),m_0} = -\tilde{h}_{i\tau}^{(m_0+v)N} V_{n-1}^{m_0-1}(\upsilon) + \sum_{p=-\infty}^{+\infty} \tilde{f}_{i\tau}^{(p+\upsilon)N} y_p^{(2),m_0} V_{n+1}^{p-1}(\upsilon) + y_p^{(2),m_0} P_n(\upsilon) \tag{15}$$

$$n = \pm 1, \pm 2, \pm 3,$$

$$\upsilon = \cos(\pi d / l).$$

The matrix operator of this set is quite continuous that allows to reduce the infinite system of linear equations and to obtain its numerical solution for the arbitrary parameters of the problem. In some specific cases with the cone structure (for example, when the number of slots is large ant their width is small compared to the period of the cone) the matrix operator is a compressive one that allows to obtain an analytical solution to the set (14), (15) using the consequent approximations method and to put down the analytical solution to the problem of electrodynamics.

ANALYTICAL SOLUTION

In the event of a large number of slots, which are sufficiently narrow compared to the period of the cone structure, under the condition of existence of the limit

$$Q = \lim_{\substack{N \to \infty \\ d/l \to 0}} \left[-\frac{1}{N} \ln \sin \frac{\pi d}{2l} \right] \tag{16}$$

the solution to the system (14), (15) is found by the method of consequent approximations and the coefficients $x_{m,n}^{(2)}$ are determined upon the formula

$$x_{m,n}^{(2)} = \frac{\tilde{\Phi}_{i\tau}^{(m)}}{\tilde{D}_{i\tau}^{(m)}} \frac{2Q}{2Q + \tilde{\Phi}_{i\tau}^{(m)}}, \quad \forall n \in Z,$$

$$\tilde{\Phi}_{i\tau}^{(m)} = \frac{\tilde{D}_{i\tau}^{(m)}}{1 + \varsigma^{(2)} 2\left(\tau^2 + \frac{1}{4}\right) \sin \gamma \tilde{D}_{i\tau}^{(m)}}. \tag{17}$$

In accordance with (5), (6), (17) the potential $V_1^{(2)}$ is represented in the following form

$$V_1^{(2)} = -\frac{4Q}{\pi^2} \int\limits_0^{+\infty} \tau \, sh\pi\tau \, \frac{K_{i\tau}(qr)}{\sqrt{r}} \sum\limits_{m=-\infty}^{+\infty} \frac{a_{m\tau}^{(\chi)} \hat{b}_{m\tau}^{(2)} \tilde{\Phi}_{i\tau}^{(m)}}{\tilde{D}_{i\tau}^{(m)} \left(2Q + \tilde{\Phi}_{i\tau}^{(m)} \right)} \frac{P_{-1/2+i\tau}^m (\pm \cos\theta)}{\frac{d}{d\gamma} P_{-1/2+i\tau}^m (\pm \cos\gamma)} e^{im\varphi} d\tau \cdot \quad (18)$$

Thus, we obtain the expression for the potential in the case with the cone structure formed up by a large number of cone impedance strips, the width of which is comparable with the period of the structure. The surface properties of this structure are determined by the impedance parameter $R^{(\chi)} = \varsigma^{(\chi)} (qr)^{-\alpha(\chi)}$ and the filling parameter Q. Variation of these parameters (as it is evident from (18)) can be adjusted by means of reflecting and transmitting properties of the cone structure.

From the representation (18) one can obtain the solution to the problem for particular cases of the cone surface:

1) $Q \to +\infty$; the continuous-solid cone, on the surface of which the impedance-type two-side conditions (1) are prescribed,

$$\lim\limits_{\varrho \to -\infty} \frac{2Q}{2Q + \tilde{\Phi}_{i\tau}^{(m)}} = 1,$$

$$V_1^{(2)} = -\frac{2}{\pi^2} \int\limits_0^{+\infty} \tau \, sh\pi\tau \, \frac{K_{i\tau}(qr)}{\sqrt{r}} \sum\limits_{m=-\infty}^{+\infty} a_{m\tau}^{(\chi)} \hat{b}_{m\tau}^{(2),p} \frac{\tilde{O}_{i\tau}^{(m)}}{\tilde{D}_{i\tau}^{(m)}} \frac{P_{-1/2+i\tau}^m (\pm \cos\theta)}{\frac{d}{d\gamma} P_{-1/2+i\tau}^m (\pm \cos\gamma)} e^{im\varphi} d\tau;$$

2) $\varsigma^{(2)} = 0$, $R^{(2)} = 0$; the semi-transparent cone consisting of a large number of ideally conducting strips (Q is the structure transparency parameter),

$$V_1^{(2)} = -\frac{4Q}{\pi^2} \int\limits_0^{+\infty} \tau \, sh\pi\tau \, \frac{K_{i\tau}(qr)}{\sqrt{r}} \sum\limits_{m=-\infty}^{+\infty} a_{m\tau}^{(\chi)} \hat{b}_{m\tau}^{(2),p} \frac{1}{\tilde{D}_{i\tau}^{(m)} + 2Q} \frac{P_{-1/2+i\tau}^m (\pm \cos\theta)}{\frac{d}{d\gamma} P_{-1/2+i\tau}^m (\pm \cos\gamma)} e^{im\varphi} d\tau;$$

this result is matching with the previously obtained one for the semi-transparent cone [8].

The spectrum of the boundary problem for the cone with the parameter Q (16) is determined by the radicals of the equation

$$2Q + \tilde{O}_\mu^{(m)} = 0 \, .$$

The analysis of the spectrum of the boundary problem in the case when the source is located on the axis of the structure ($\theta_0 = \pi$, $m = 0$) indicated that within the structure of the scattered field there is present a surface wave

stipulated by the availability of the impedance dependent upon the radial coordinate. The field of that wave is exponentially decreasing at increasing of the angular coordinate θ, which is equal to the angle between the cone axis and the direction to the point of observation. The possibility of existence of this type of surface waves within the structure of the field scattered by the continuous-solid impedance cone is mentioned in the paper [9].

CONCLUSION

The approach to the solution of the scalar problem wave scattering on the impedance cone surface with periodic longitudinal slots is developed in this paper. Its essence lies in using the Kontorovich-Lebedev integral transformation and the semi-inversion method. It is demonstrated that the considered problem is equivalent to the solution of the system of linear equations of the second kind with respect to Fourier coefficients of the sought function. The rigorous analytical solution to the cone structure formed by a large number of impedance strips of the cone structure is obtained. The trustworthiness of the solution is substantiated by its match with the known results in the extreme cases. The surface wave stipulated by the special type of surface impedance is detected within the structure of the field.

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