12th IMECO TC1&TC7 Joint Symposium on
Man Science & Measurement
September, 3-5, 2008, Annecy, France

UNCERTAINTY MEASUREMENT AND DYNAMIC SYSTEM CHAOTICAL BEHAVIOUR

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Abstract Chaotic behaviour of nonlinear dynamic system as a mechanism of influence on uncertainty measurement are discussed. Based on the logistic equation as a mathematical models of measeurement uncertainty investigated as a function of state of dynamic system. In the case when dynamic chaos take place in system, uncertainty measurement are increasing and has anormal distribution.

Keywords: uncertainty, dynamic chaos, logistic equation.

1. INTRODUCTION

According to GUM, measurands of static objects or dynamic system must be described by well-defined physical quantities, represented by the essentially unique values [1]. Physical models allow usage of well-known statistical methods for analysis of measurement results and estimation uncertainty measurement.

In case systems with deterministic behavior we have a type of well-investigated dynamic measurements. For dynamic systems (physical or chemical) with chaotic behavior (as example strange attractor [2]) estimation measurement results must to take into account the character of dynamical changes [3, 4]. Now standard mathematical and physical methods, which need for measurement results estimations, do not permit investigation influence of chaotic behavior dynamical systems on uncertainty measurement.

This report present investigations of mathematical models, which take into account deterministic and chaotic solutions, on conditions of estimations uncertainty measurement.

2. PURPOSE

Main arm of this work direct on investigation of uncertainty transformation under condition that time changes of measurand value have a behavior, connected with chaotic dynamic in system.

For realization this arm in paper was using mathematical model based on nonlinear discrete difference equation that are well-known in literature [5]. This equation have some different types of solutions. One type of this solutions, describes deterministic periodical behavior in time, while other type of solutions describes chaotic or non-regular behavior. It was necessary to make clear how statistical properties of chaotically solutions influence on value uncertainty.

As a GUM gives no recommendations how to estimate uncertainty measurement in the dynamic systems with a chaotic behavior, another purpose present paper are to investigate how uncertainty of equation parameter value influence on dynamic chaos statistical properties. And then how uncertainty measurement depend from properties.

Last arm of this paper – to study connection between time of transformations in dynamic system and time between individual measurements.

3. METHOD

GUM establishes that the measurand X is defined if the functional relationship or the measurement model is given

$$X = F(\vec{\mu}) , \tag{1}$$

where $\vec{\mu} = (\mu_1, \ldots \mu_N)$ is a N -dimension vector of input quantities.

In dynamic systems, the measurement model is given by a system of differential equations [6]

$$\frac{\partial \vec{x}}{\partial t} = f(\vec{x}, \vec{\mu}) , \tag{2}$$

where $\vec{x}$ is a vector of measurands, $\vec{\mu}$ is a vector of input quantities and $f(\vec{x}, \vec{\mu})$ - nonlinear function.

In accordance to GUM, the measurement of quantity x as a single (scalar) output quantity has physical sense in case stable and stationary state, i.e.

$$\frac{\partial x}{\partial t} = 0 . \tag{3}$$

Measurement model is a nonlinear algebraic equation

$$f(x, \mu) = 0 , \tag{4}$$

or more simple equation [1]

$$x - f(\mu) = 0. \quad (5)$$

For describing dynamic system's behavior in report, based on (2) was using discrete difference equation [5] that is well known as the logistic mapping equation

$$x_{n+1} = \mu x_n (1 - x_n). \quad (6)$$

Solutions of (6), corresponding to actual states of a physical system, are attracting and limiting points or cluster sets. Only the limit of a sequence x_n presents the practical interest for measurement

$$x(\mu) = \lim_{n \to \infty} x_n(\mu). \quad (7)$$

Accordingly, in the result of measuring we determine the numerical value of $x(\mu)$, which depends on parameter μ. So, all cases of the measurement models are based on (7) and we have to measure μ for definition of $x(\mu)$. The solutions (7) of the equation (6) are well investigated (see, for example [5]), therefore, in the present consideration these solutions were treated as the measuring equation for $x(\mu)$ at different values of μ.

4. RESULTS

Basing on these equations, which have different physical senses, the estimation of probable statistically- spreaded measurement results were carried out for different values of μ. Because of the sensitivity of values $x(\mu)$ to values of μ, the basic measurement problem, which is necessary to solve, is defining the conditions, at which the system behavior influence the statistical spreading of measurement results.

1) $0 < \mu \le 1$. The solution (6) represents an inconvertible attractive fixed point $x = 0$. In this case the measuring of value of quantity x presents a trivial problem.

2) $1 < \mu \le 3$. The square-law mapping has a unique fixed limiting attractive point $x^{(1)}$, which value is determined by the expression

$$x^{(1)} = 1 - 1/\mu, \quad (8)$$

for any initial value x_0.

The mean value $\overline{x^{(1)}}$ and the standard deviation $u_c(x^{(1)})$ can be expressed as follows:

$$\overline{x^{(1)}} = \frac{1}{n} \sum_{k=1}^{n} (1 - 1/\mu_k), \quad (9)$$

$$u_c(x^{(1)}) = 1/\mu^2 u_c(\mu), \quad (10)$$

where

$$u_c(\mu) = \sqrt{\frac{1}{n(n-1)} \sum_{k=1}^{n} (\mu_k - \overline{\mu})^2} \quad (11)$$

4) $3.5699 < \mu \le 4$. Under these values of the μ parameter, the limit (9) is cluster set, which sizes are determined by the region of existence of the solutions of this equation. The structure of the set is determined by a value of the μ parameter. In [2] it is shown, that inside a unit square of a region of definition of quantity x_n, there is a random change of values of quantity x_n. It means that every next iteration gives value of quantity x_n, which randomly differs from previous value of quantity x_{n-1}. It means, that at any initial condition x_0 the result of the sequentially fulfilled iterations of equation (5), when $n \to \infty$ is the sequence of random values x_n. On the other hand, the existence of such solution of equation (5) indicates the absence of the determinate measuring equation in the sense, in which the equation (5) did serve at other values μ.

Thus, the solution of considered square-law mapping becomes not an inconvertible limit point characterized by a unique value, but range of values x_n, in which the random values x_n are defined by a probability frequency function - $\varphi(x_n, \mu)$. The properties of the square-law mapping are such that at any value of μ_i the probability density function $\varphi(x_n, \mu_i)$ is shaped. At great many enough of iterations the distribution of values of quantity x_n becomes practically continuous that allows proceed to continuous probability density function - $\varphi_i(x, \mu_i)$.

At each value of μ_i the mean value $\overline{x_i}$ will be determined by the relevant probability density function

$$\overline{x_i} = \int_0^1 x \varphi_i(x, \mu_i) dx. \quad (12)$$

The feature of this equation is that the value $\overline{x_i}$ depends on sort of a function φ_i which, in turn, exhibits strong sensitivity to value of the μ_i parameter. That is, the small change in value of this parameter gives an essential change to function $\varphi_i(x, \mu_i)$ character, that in turn, gives an essential change in value $\overline{x_i}$. It means that values of difference $\overline{x_i} - \overline{x_j}$ at close values of μ_i and μ_j can differ

by an arbitrary random amount. Normal law of spreading of the apparent parameter's μ values, the variance $\overline{x_i}$ values will be anomalously large.

If m of the observations of the quantity μ_i are carried out, the mean value of the result of measuring of quantity $\bar{x}$ is determined by the expression

$$\bar{x} = \frac{1}{m}\sum_{i=1}^{m}\overline{x_i} = \frac{1}{m}\sum_{i=1}^{m}\int_0^1 x\varphi_i(x,\mu_i)dx. \qquad (13)$$

In the result of measuring stipulated by the spreading of measured μ values, the variance is determined by the expression

$$u_c^2 = \frac{1}{n(n-1)}\sum_{i=1}^{m}(\overline{x_i} - \bar{x})^2 \qquad (14)$$

Thus, the result of observation of the statistical spreading of values of driving parameter μ is magnified by random dynamics of the square-law mapping and this gives abnormal magnification of the uncertainty $\bar{x}$.

With the understanding what gives the acknowledgement of the random behavior of the dynamic system at $n \to \infty$, we shall consider more in details sampled iterations of the mapping (5). Thus, the possible conditions of the practical implementation of the process of measuring of quantity x_n depend on an interval τ between observations of the quantities x_n and number of observations N. If T is an interval between sequential iterations of the equation (5), it is feasible to consider two extreme cases: first is for $N\tau \le T$, and second is for $N\tau > T$.

If the condition $N\tau \le T$ is satisfied, the mean value $\overline{x_n}$ with standard deviation $u_c(x_n)$ will be set on n-th iteration of square-law mapping which is stipulated with the uncertainty $u_c(\mu)$. At the next iteration of mapping the other value of x_{n+1} will be measured, which also will be carried out with the uncertainty $u_c(\mu)$. And each time, after the next iteration of the mapping, it will be measured randomly modified value of x_n. Nobody can ensure the reproducibility of measuring in the square-law mapping operating in a random mode since the physically stable values of observed data do not exist. Therefore, a measure uncertainty is necessary for observed data generated randomly within a region of finite values.

If the interval τ between measuring of the quantity x_n is more than a time slice T, the series of random values x_n will be captured in result of N observations. Thus, as it was shown above, the uncertainty of the result, both single observation and result of measuring, will be stipulated by the stochastic dynamics of the behavior of the measurand. As the region of random change of quantity x_n is restricted, it is possible to view this region as limiting, stable borders of explored system, which "is spread" statistically in finite region.

For an example we will consider a random sequence of values x_n, generated by the square-law mapping (8) at μ =4. In [5] the statistical property of the sequence is characterized by a probability density distribution function $P_4(x)$ (coefficient 4 points at parameter value μ =4)

$$P_4(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}. \qquad (15)$$

This function is defined in an interval of values $0 < x < 1$. It was taken into account that the stochastic process implemented at μ =4 with intermixing and exponential divergence of close trajectories [5], is ergodic.

In the theory of errors such arc-sinusoidal function is well-known as a distribution function of errors for electrical methods of the measuring, which have interference from power lines with frequency 50 Hz. In a reviewed case the distribution function takes into account the random character of behavior of the dynamic system. Utilizing function (15), it is possible to calculate the mean value of the dynamic quantity $\overline{x_4}$

$$\overline{x_4} = \frac{1}{\pi}\int_0^1 \sqrt{\frac{x}{1-x}}dx = \frac{1}{2}, \qquad (16)$$

and the standard deviation

$$S = \sqrt{\frac{1}{\pi}\int_0^1 \frac{(x-1/2)^2}{\sqrt{x(1-x)}}dx} = 0{,}353. \qquad (17)$$

Thus, even at the usage of an absolutely precise parameter's value μ (in our case μ =4), the one-dimensional mapping generates a random sequence of x_n, which is characterized by a probability density distribution function (12) that helps to calculate the mean value and the standard deviation.

From this it follows that the existence of a random mode in behavior of dynamic system is a key factor making that the stationary unique value for a measurand simply does not exist. Therefore, the uncertainty of measuring of the system parameters, which is taking place in a random mode,

is stipulated by a random spreading of the values of a measurand.

5. DISCUSSION AND CONCLUSIONS

Utilizing one-dimensional discrete mapping for exposition of the character of the behavior of dynamic system, it is shown how the stochastic behavior of the explored dynamic system can influence an estimation of system parameters from the observed data. It is shown that the statistical character of the behavior of the dynamic system is the decisive factor for estimation of the parameters of the system from the observed data. On a theoretical example of one-dimensional logistic mapping it is shown the necessity of the acknowledgement of the random behavior of the dynamic system for the estimation of the observed data.

Approach introduced in the present examination is one of the perspective ways in the development of the theory of measuring, including methods of uncertainty estimation or an error estimation of the carried-out measurements. It is possible to formulate a blanket problem, which solution seems to be extremely in need now.

Examinations applied to complex dynamic systems (such as different sort biological objects, social systems, objects with a composite interior kinetics of chemical reactions, and also all systems based on the thermodynamic non-equilibrium processes [7-8]) should take into account the fundamental synergetic effects (such as self-organizing resulting in the development of the spatial, time and time-multiplexed structures and processes of the dynamic chaos causing irregular, random behavior of the dynamic systems), explicating the behavior of these systems.

Today, the explicated methods of the experimental examinations of dynamic chaos are not methods for estimation of the observed data, which ensure unity of measuring. And for this reason, the examinations of the influence of the dynamic chaos in the dynamic systems on the observed data should have a metrological bias, which can lead to development of the uniform approach to problem of measuring.

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