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TECHNICAL PHYSICS

Unsteady Diffraction by a Nonclosed Cone

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A method for solving a stationary problem of wave scattering by a semi-infinite circular cone with slots periodically cut along its generating lines is presented in [1]. In this paper, we pioneer the presentation of an algorithm for constructing an unsteady Green's function of a cone with longitudinal slots.

FORMULATION AND THE METHOD OF SOLUTION OF THE PROBLEM

The conic structure under consideration represents a perfectly conducting semi-infinite thin cone with N slots periodically cut along its generatrices. We introduce a spherical coordinate system r, ϑ, φ with the origin at the vertex of the cone; its axis coincides with the oz-axis of the Cartesian coordinate system (see the figure). Furthermore, we use the following notation: 2γ is the cone opening angle, $l = \frac{2\pi}{N}$ is the structure period, and d is the slot width (d and l are the values of the dihedral angles formed by intersections of planes drawn through the cone axis and edges of conic strips). In the coordinate system chosen, the cone is determined by a set of points:

$$\Sigma = \{(r, \vartheta, \varphi) \in R^3:\ r \in [0, +\infty),\ \vartheta = \gamma,\ \varphi \in L\},$$

where

$$L = \bigcup_{p=1}^{N} L_p, \quad L_p = \left((p-1)l + \frac{d}{2},\ pl - \frac{d}{2}\right),$$

$$CL = [0, 2\pi)\backslash L.$$

The desired Green's function $G(\mathbf{r}, \mathbf{r}_0, t, t_0)$ satisfies

(*1*) the equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}, \mathbf{r}_0, t, t_0) = -\delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0);$$

(*2*) the boundary condition at the cone strips

$$G(\mathbf{r}, \mathbf{r}_0, t, t_0)|_\Sigma = 0; \text{ and}$$

(*3*) the initial conditions

$$G = \frac{\partial G}{\partial t} \equiv 0 \quad \text{for} \quad t < t_0.$$

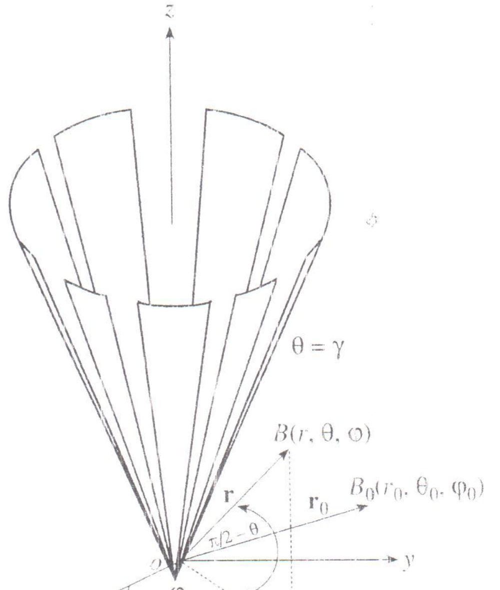
We represent the function G in the form

$$G(\mathbf{r}, \mathbf{r}_0, t, t_0) = G_0(\mathbf{r}, \mathbf{r}_0, t, t_0) + G_1(\mathbf{r}, \mathbf{r}_0, t, t_0),$$

$$G_0(\mathbf{r}, \mathbf{r}_0, t, t_0) = \frac{\delta(\hat{t} - R/c)}{4\pi R}. \tag{1}$$

$$\hat{t} = t - t_0, \quad R = |\mathbf{r} - \mathbf{r}_0|.$$

Here, c is the speed of light; the function G_1 is conditioned by the presence of the cone surface. For find-



Geometry of the problem.

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ing G, we use the Laplace transformation

$$G^s(\mathbf{r}, \mathbf{r}_0, t_0) = \int_0^{+\infty} G(\mathbf{r}, \mathbf{r}_0, t, t_0) e^{-st} dt, \quad s > 0, \tag{2}$$

where G^s is the Green's function for the corresponding steady boundary value problem. This function satisfies the inhomogeneous Helmholtz equation, the first boundary condition on the cone strips, the condition at infinity in space, and the condition near the boundary irregularities (the cone vertex and the strip edges).

In accordance with (1),

$$G^s(\mathbf{r}, \mathbf{r}_0, t, t_0) = G_0^s(\mathbf{r}, \mathbf{r}_0, t, t_0) + G_1^s(\mathbf{r}, \mathbf{r}_0, t, t_0),$$

$$G_0^s(\mathbf{r}, \mathbf{r}_0, t, t_0) = e^{-st_0} \frac{e^{-qR}}{4\pi R}, \quad q = \frac{s}{c}.$$

To solve the steady problem, we invoke the Kontorovich–Lebedev integral transformation

$$F(\tau) = \int_0^{+\infty} f(r) \frac{K_{i\tau}(qr)}{\sqrt{r}} dr, \tag{3}$$

$$f(r) = \frac{2}{\pi^2} \int_0^{+\infty} \tau \sinh \pi\tau F(\tau) \frac{K_{i\tau}(qr)}{\sqrt{r}} d\tau. \tag{4}$$

Here, $K_\mu(z)$ is the Macdonald function. Taking into account the representation

$$G_0^s = \frac{2}{\pi^2} \int_0^{+\infty} \tau \sinh \pi\tau \sum_{m=-\infty}^{+\infty} a_{m\tau}^s U_{m\tau}^{(0)} e^{im\varphi} \frac{K_{i\tau}(qr)}{\sqrt{r}} d\tau,$$

$$U_{m\tau}^{(0)}(\vartheta, \vartheta_0) = \begin{cases} P_{-1/2+i\tau}^m(\cos\vartheta) P_{-1/2+i\tau}^m(-\cos\vartheta_0), & \vartheta < \vartheta_0 \\ P_{-1/2-i\tau}^m(-\cos\vartheta) P_{-1/2+i\tau}^m(\cos\vartheta_0), & \vartheta_0 < \vartheta, \end{cases}$$

we seek the function G_1^s in the form of Kontorovich–Lebedev integral (3), (4):

$$G_1^s = \frac{2}{\pi^2} \int_0^{+\infty} \tau \sinh \pi\tau \sum_{m=-\infty}^{+\infty} b_{m\tau}^s U_{m\tau}^{(1)} \frac{K_{i\tau}(qr)}{\sqrt{r}} d\tau, \tag{5}$$

$$U_{m\tau}^{(1)} = \sum_{n=-\infty}^{+\infty} x_{m,n+m_0}(\tau) \frac{P_{-1/2+i\tau}^{m+nN}(\pm\cos\vartheta)}{P_{-1/2+i\tau}^{m+nN}(\pm\cos\gamma)} e^{i(m+nN)\varphi}. \tag{6}$$

Here, $P_\mu^m(\cos\vartheta)$ is the associated Legendre function of the first kind; $a_{\nu\tau}^s$ and $b_{\nu\tau}^s$ are known coefficients, and x_m and x_n are unknown coefficients; the superscripts of the Legendre-function arguments in (6) correspond to the region $0 < \vartheta < \gamma$, and the subscripts correspond to the region $\gamma < \vartheta < \pi$; and $\frac{m}{N} = \nu + m_0$, $-\frac{1}{2} \le \nu < \frac{1}{2}$, m_0 being the integer closest to $\frac{m}{N}$. As a result of using the boundary condition, the conjunction condition for G_1^s in the slots, and the semiconversion method [1], the steady problem is reduced to the solution of an infinite set of linear Fredholm-type algebraic equations of the second kind with respect to the coefficients $y_{m,n}$ related to $x_{m,n}$:

$$A_\nu(u) y_{m,0} = V^{m_0}(u) + \sum_{p=-\infty}^{+\infty} \frac{|p|}{p} \varepsilon_{m,p} V^p(u) y_{m,p}, \tag{7}$$

$$y_{m,n} = V_{n-1}^{m_0-1}(u) + \sum_{p=-\infty}^{+\infty} \frac{|p|}{p} \varepsilon_{m,p} y_{m,p} V_{n-1}^{p-1}(u) + y_{m,0} P_n(u), \quad n \neq 0, \tag{8}$$

$$y_{m,n} = (-1)^{n-m_0} \frac{n+\nu}{m_0+\nu} \frac{|n|}{n} \frac{x_{m,n}}{1-\varepsilon_{m,n}},$$

$$A_\nu(u) = \frac{2P_{\nu-1}(-u)}{\nu(P_{\nu-1}(-u) + P_\nu(-u))},$$

$$u = \cos\delta, \quad \delta = \frac{l-d}{d}\pi,$$

where $V^m(u)$ and $V_{n-1}^{p-1}(u)$ are known functions [1]. For the matrix elements $\varepsilon_{m,n}$ of the system, in the case of $N(n+\nu) \gg 1$, the estimate holds:

$$\varepsilon_{m,n} = O\left(\frac{\sin^2\gamma}{N^2(n+\nu)^2}\right).$$

It should be emphasized that the desired coefficients $y_{m,n}$ (and, consequently, $x_{m,n}$) are independent of parameter q. This is essential for conversing and solving the unsteady problem. A solution to the system of equations (7), (8) exists and is unique. For arbitrary parameters of the problem, this solution can be obtained by the reduction method. In the case of a semitransparent cone, a cone with narrow slots, and a narrow cone, the norm of the system matrix operator is smaller than unity. This allows the method of successive approximations to be employed for solving the system of equations. Using the procedure of a steady-problem conversion for a continuous cone [2], we obtain representations for the Green's function G_1 in the case of a cone with longitudinal slots:

$$G_1 = \frac{c}{4\pi r r_0}\eta\left(\hat{t} - \frac{r+r_0}{c}\right)\sum_{m=-\infty}^{+\infty} e^{im\varphi} \tag{9}$$

$$\times \int_0^{+\infty} g_{m\tau} U_{m\tau}^{(1)} P_{-1/2+i\tau}(\cosh b)\, d\tau, \quad \gamma < \vartheta_0.$$

Here,

$$g_{m\tau} = (-1)^{m+1} e^{-im\varphi_0}\tau\tanh\pi\tau\frac{\Gamma(1/2-m+i\tau)}{\Gamma(1/2+m+i\tau)}$$

$$\times P_{-1/2+i\tau}^{m}(-\cos\vartheta_0)P_{-1/2+i\tau}^{m}(\cos\gamma),$$

$$\cosh b(\hat{t}) = \frac{\hat{t}^2c^2 - r^2 - r_0^2}{2rr_0},$$

$\eta(x)$ is the Heaviside function, and $\Gamma(z)$ is the gamma-function.

REPRESENTATIONS FOR THE GREEN'S FUNCTION

Semitransparent cone. In the case of a semitransparent cone determined by the existence of the limit

$$\lim_{\substack{N \to +\infty \\ d/l \to 1 \\ (\delta \to 0)}} \left[\frac{2}{N}\ln\frac{2}{\delta}\right] = Q > 0,$$

we obtain from (9) the following integral representations for the unsteady Green's function:

$$G_1 = \frac{c}{4\pi r r_0}\eta\left(\hat{t} - \frac{r+r_0}{c}\right)\sum_{m=-\infty}^{+\infty} e^{im\varphi}\int_0^{+\infty} w_{m\tau} \tag{10}$$

$$\times P_{-1/2+i\tau}^{m}(\cos\vartheta)P_{-1/2+i\tau}(\cosh b)\,d\tau, \quad 0 < \vartheta < \gamma,$$

$$G_1 = \frac{c}{4\pi r r_0}\eta\left(\hat{t} - \frac{r+r_0}{c}\right)\sum_{m=-\infty}^{+\infty} e^{im\varphi}\int_0^{+\infty} w_{m\tau} \tag{11}$$

$$\times \frac{P_{-1/2+i\tau}^{m}(\cos\gamma)}{P_{-1/2+i\tau}^{m}(-\cos\gamma)}P_{-1/2+i\tau}^{m}(-\cos\vartheta)P_{-1/2+i\tau}(\cosh b)\,d\tau,$$

$$\gamma < \vartheta < \pi,$$

$$w_{m\tau} = \frac{(1-\varepsilon_{m,0})g_{m\tau}}{(1-\varepsilon_{m,0}) + 2mQ}.$$

If the source is placed on the cone axis ($\vartheta_0 = \pi$, $\varphi_0 = 0$; $m = 0$), expressions (10) and (11) are simplified:

$$G_1 = -\frac{c}{4\pi r r_0}\eta\left(\hat{t} - \frac{r+r_0}{c}\right)\int_0^{+\infty}\frac{\tau\tanh\pi\tau}{D_{i\tau}}P_{-1/2+i\tau}(\cos\gamma)$$

$$\times P_{-1/2+i\tau}(-\cos\gamma)P_{-1/2+i\tau}(\cos\vartheta)P_{-1/2+i\tau}(\cosh b)\,d\tau,$$

$$0 < \vartheta < \gamma,$$

$$G_1 = -\frac{c}{4\pi r r_0}\eta\left(\hat{t} - \frac{r+r_0}{c}\right)\int_0^{+\infty}\frac{\tau\tanh\pi\tau}{D_{i\tau}}[P_{-1/2+i\tau}(\cos\gamma)]^2 \tag{12}$$

$$\times P_{-1/2+i\tau}(-\cos\vartheta)P_{-1/2+i\tau}(\cosh b)\,d\tau, \quad \gamma < \vartheta < \pi,$$

$$D_{i\tau} = \pi P_{-1/2+i\tau}(\cos\gamma)P_{-1/2+i\tau}(-\cos\gamma) + 2Q\cosh\pi\tau.$$

Passing to integrating over the imaginary axis ($\mu = i\tau$) in (12) and using the fundamental residue theorem, we arrive at a representation for G_1 in the form of a series:

$$G_1 = \frac{c}{2\pi r r_0}\sum_{j=0}^{+\infty}\frac{\mu_j}{\frac{d}{d\mu}D_{\mu_j}}P_{-1/2+\mu_j}(\cos\gamma)$$

$$\times P_{-1/2+\mu_j}(-\cos\gamma)P_{-1/2+\mu_j}(\cos\vartheta)Q_{-1/2+\mu_j}(\cosh b), \tag{13}$$

$$0 < \vartheta < \gamma, \quad c\hat{t} > r + r_0, \quad D_{\mu_j} = 0,$$

where $Q_\zeta(z)$ is the Legendre function of the second kind. Thus, the spectrum for the unsteady boundary value problem is the same as that for the corresponding steady one; it depends on the angular parameters of the conic structure [1]. In the particular case of a semitransparent cone ($Q \gg 1$), the spectrum is determined by the set $\{\mu_j\}_0^{+\infty}$:

$$\mu_j = \frac{1}{2} + j + \frac{1}{2Q}[P_j(\cos\gamma)]^2 + O(Q^{-2}),$$

$$j = 0, 1, 2, \ldots$$

In the case of a steady-state mode ($\hat{t} \gg 1$), we can restrict ourselves to approximation (13) for G_1:

$$G_1 \sim \frac{c}{2\pi r r_0}\frac{\mu_0}{\frac{d}{d\mu}D_{\mu_0}}P_{-1/2+\mu_0}(\cos\gamma)P_{-1/2+\mu_0}(-\cos\gamma)$$

$$\times P_{-1/2+\mu_0}(\cos\vartheta)Q_{-1/2-\mu_0}\left(\frac{\hat{t}^2c^2 - r^2 - r_0^2}{2rr_0}\right).$$

For a semitransparent cone with the filling parameter $Q \gg 1$ and $\hat{t} \gg 1$,

$$\mu_0 = \frac{1}{2} + \frac{1}{2Q} + O(Q^{-2}), \quad G_1 \sim -\frac{c}{4\pi^2 Q}(\hat{t}c)^{-2},$$

$$0 < \vartheta < \gamma$$

under the condition that $\frac{1}{Q}|\ln(0.5\sin\gamma)| \ll 1$.

Narrow slots. In the case of a cone with narrow slots $\left(\frac{d}{l} \ll 1;\ 1 + u \ll 1\right)$, the asymptotic expansion of the Green's function in terms of the parameter $1 + u$, which holds far from the slots, has the form ($\vartheta_0 = \pi$, $\varphi_0 = 0$)

$$G_1 = -\frac{c}{4rr_0}\int_0^{+\infty}\frac{\tau\tanh\pi\tau}{\cosh\pi\tau}\frac{[P_{-1/2+i\tau}(\cos\gamma)]^2}{\Omega_\tau}$$

$$\times P_{-1/2+i\tau}(-\cos\vartheta)P_{-1/2+i\tau}(\cosh b)d\tau + \frac{c}{8\pi rr_0}\frac{1+u}{N}$$

$$\times\sum_{n\neq 0}e^{inN\varphi}\int_0^{+\infty}\tau\tanh\pi\tau\frac{\widehat{W}_\tau P_{-1/2+i\tau}(\cos\gamma)}{\Phi_\tau}$$

$$\times\frac{P^{nN}_{-1/2+i\tau}(-\cos\vartheta)}{P^{nN}_{-1/2+i\tau}(-\cos\gamma)}P_{-1/2+i\tau}(\cosh b)d\tau$$

$$+ O((1+u)^2\ln(1+u)),$$

$$\gamma < \vartheta < \pi,$$

$$\Omega_\tau = B_\tau - \frac{1}{N}\ln\frac{1-u}{2}, \quad \Phi_\tau = \widehat{W}_\tau + \frac{1}{N}\frac{1+u}{2}(B_\tau + \widehat{W}_\tau),$$

$$B_\tau = \frac{\pi}{\cosh\pi\tau}P_{-1/2+i\tau}(\cos\gamma)P_{-1/2+i\tau}(-\cos\gamma),$$

$$\widehat{W}_\tau = \frac{1-\varepsilon_{0,n}}{N|n|\varepsilon_{0,n}}.$$

A similar representation also holds in the case of $0 < \vartheta < \gamma$. The spectrum of eigenvalues is determined by the roots of the equation with a small right-hand side:

$$\frac{\pi}{\cos\pi\mu}P_{-1/2+\mu_k^\pm}(\cos\gamma)P_{-1/2+\mu_k^\pm}(-\cos\gamma) = \frac{1}{N}\ln\frac{1-u}{2},$$

where

$$\mu_k^\pm = \alpha_k^\pm - \frac{1+u}{2N}$$

$$\times\frac{\cos\pi\mu}{\pi\frac{d}{d\mu}[P_{-1/2+\mu}(\cos\gamma)P_{-1/2+\mu}(-\cos\gamma)]\big|_{\mu=\alpha_k^\pm}}$$

$$+ O((1+u)^2),$$

$$P_{-1/2+\alpha_k^+}(\cos\gamma) = 0, \quad P_{-1/2+\alpha_k^-}(-\cos\gamma) = 0,$$

$$k = 0, 1, 2, \ldots$$

In the limiting case of vanishing slots ($d \longrightarrow 0$, $u \longrightarrow -1$), the expressions obtained coincide with the results for the continuous cone [2].

Narrow cone. In the case of a narrow cone ($\gamma \ll 1$), the asymptotic representation for G_1 has the form ($\vartheta_0 = \pi$, $\varphi_0 = 0$)

$$G_1 = G_1^{\mathrm{istr}} + G_1^{\mathrm{slt}} + O\left(\ln^{-3}\left(\frac{2}{\gamma}\right)\right), \quad \gamma < \vartheta < \pi. \qquad (14)$$

Here,

$$G_1^{\mathrm{istr}} = \frac{c}{8\pi rr_0}\frac{1}{\ln(2/\gamma)}\eta\left(\bar{t} - \frac{r+r_0}{c}\right)$$

$$\times\left[\frac{1}{\pi(\cosh b - \cos\vartheta)} - \Phi_1(\bar{t}, r, r_0, \vartheta)\right]$$

is the asymptotic form of the Green's function for the continuous cone:

$$\Phi_1 = \frac{1}{2}\int_0^{+\infty}\frac{\tau\tanh\pi\tau}{\cosh\pi\tau}\left[\Psi\left(-\frac{1}{2}+i\tau\right) + C - \frac{1}{2}\pi i\tanh\pi\tau\right]$$

$$\times P_{-1/2+i\tau}(-\cos\vartheta)P_{-1/2+i\tau}(\cosh b)d\tau,$$

where $\Psi(z)$ is the psi-function, C is the Euler constant, and the term

$$G_1^{\mathrm{slt}} = -\frac{c}{8\pi rr_0}\frac{1}{\ln^2(2/\gamma)}\frac{1}{N}\ln\left(\frac{1-u}{2}\right)\frac{1}{\cosh\pi\tau - \cos\vartheta}$$

accounts for the effect of the slots. Representation (14) for G_1 is valid far from the slots and the cone vertex.

CONCLUSIONS

In this paper, we proposed and substantiated an algorithm for constructing an unsteady Green's function of a first boundary value problem for a semi-infinite circular cone with periodic longitudinal slots. The problem of finding the Green's function for the wave equation was reduced to solving a system of linear algebraic equations with respect to the Fourier coefficients of the desired function. For the cases of a semi-transparent cone, a cone with narrow slots, and a narrow cone, both integral representations and representations in the form of a series are obtained for the Green's function. The spectrum of the unsteady first boundary value problem is shown to be the same as that for the corresponding steady one. The algorithm proposed can be used in solving boundary value problems with a more complicated conic geometry.

The results of this study were reported in part at the Second International Conference "Urgent Problems of Computational Physics" (July 24–29, 2000, Dubna, Russia) [3].

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