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NONSTATIONARY EXCITATION OF A CONE WITH SLOTS

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Abstract

The time-dependent Green's function boundary problem for a semi-infinite circular perfectly conducting cone with periodical longitudinal slots is considered. This geometry can be regarded as a model of conical slotted antennas. The solution method employs Laplace inversion, the Kontorovich-Lebedev integral transforms and the Riemann-Hilbert method. Representations for the scalar Green's functions for some particular cases of the structure and time parameter are derived.

1. INTRODUCTION

The dyadic Green's function is a very useful tool for solving boundary problems to investigate electrodynamics characteristics of complex structures. Now the interest to cones and slot structures has been raised up because of them applications at the antennas techniques and radiolocaton. The transient boundary problem solution for an isotropic cone is already associated with mathematical difficulties [1]. One should use more complicated methods to solve the stationary problem for a cone with longitudinal slots [2]. The purpose of this study is to find representations for the time-dependent Green's function for a perfectly conducting cone with periodical longitudinal slots. The structure under consideration can be regarded as a suitable model of a slotted cone antenna with controlled beams and field polarization.

2. PROBLEM STATEMENT AND METHOD

The structure under consideration is a semi-infinite circular perfectly conducting cone with N slots cut along rulings (longitudinal slots). The geometry of the configuration and the assumed spherical coordinate system r, ϑ, φ are shown in Fig.1. In this coordinate system the cone is defined by an equation $\vartheta = \gamma$. The structure period $l = 2\pi / N$ and the slot width of the cone d are angular values. The slot width is a value of the dihedral angle formed by planes those pass through the cone axis and cone strip edges.

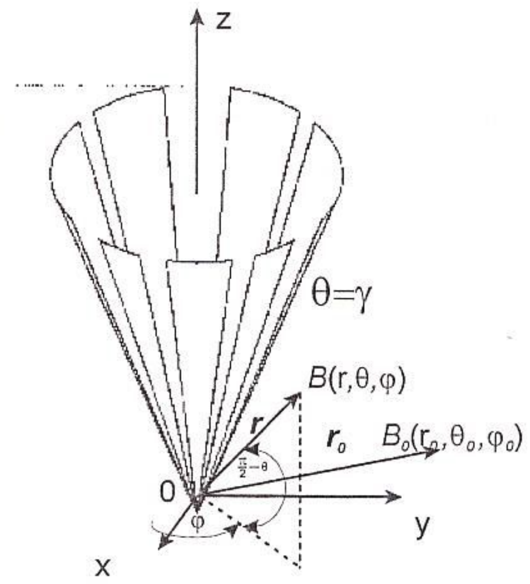

Figure 1.

The time-dependent Green's function $G(\vec{r},\vec{r}_0,t,t_0)$ satisfies:

1) the partial differential equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\vec{r},\vec{r}_0,t,t_0) = -\delta(\vec{r}-\vec{r}_0)\delta(t-t_0), \tag{1}$$

2) the boundary condition

$$G(\vec{r},\vec{r}_0,t,t_0) = 0 \text{, at } \vartheta = \gamma, \tag{2}$$

3) the initial conditions

$$G = \frac{\partial G}{\partial t} \equiv 0 \text{, at } t < t_0. \tag{3}$$

The solution for $G(\vec{r},\vec{r}_0,t,t_0)$ is written as the sum of a free-space field $G_0(\vec{r},\vec{r}_0,t,t_0)$ and a scattered field $G_1(\vec{r},\vec{r}_0,t,t_0)$, due to the presence of the cone

$$G(\vec{r},\vec{r}_0,t,t_0) = G_0(\vec{r},\vec{r}_0,t,t_0) + G_1(\vec{r},\vec{r}_0,t,t_0), \tag{4}$$

where $G_0(\vec{r},\vec{r}_0,t,t_0) = \dfrac{\delta\left[t - t_0 - |\vec{r}-\vec{r}_0|/c\right]}{4\pi|\vec{r}-\vec{r}_0|}$,

c is speed of light in the medium surrounding the cone. The time-dependent Green's function $G(\vec{r},\vec{r}_0,t,t_0)$ can be obtained via the Laplace inversion from the time-harmonic Green's function $G^s(\vec{r},\vec{r}_0,t_0)$

$$G^s(\vec{r},\vec{r}_0,t_0) = \int_0^{+\infty} G(\vec{r},\vec{r}_0,t,t_0)e^{-st}dt, s > 0 \tag{5}$$

that satisfies the three-dimensional wave equation, the Dirichlet boundary condition, radiation condition at $r \to \infty$, singularity conditions at the tip and slot edges. Accoding to (4),

$$G^s(\vec{r},\vec{r}_0,t,t_0) = G_0{}^s(\vec{r},\vec{r}_0,t,t_0) + G^s{}_1(\vec{r},\vec{r}_0,t,t_0), \tag{6}$$

where

$$G_0{}^s(\vec{r},\vec{r}_0,t,t_0) = e^{-st_0}\frac{e^{-q|\vec{r}-\vec{r}_0|}}{4\pi|\vec{r}-\vec{r}_0|}, \; q = \frac{s}{c}. \tag{7}$$

The method for solving the stationary boundary problem for G^s uses the Kontorovich-Lebedev integral transforms

$$F(\tau) = \int_0^{+\infty} f(r)\frac{K_{i\tau}(\beta r)}{\sqrt{r}}dr, \tag{8}$$

$$f(r) = \frac{2}{\pi^2}\int_0^{+\infty} \tau \sinh \pi\tau F(\tau)\frac{K_{i\tau}(\beta r)}{\sqrt{r}}d\tau, \tag{9}$$

here $K_\mu(z)$ is the Macdonald function. The function G_0^s can be represented in terms of the Kontorovich-Lebedev transform (9), [3]

$$G_0^s = \frac{2}{\pi^2}\int_0^{+\infty} \tau sh\pi\tau \sum_{m=-\infty}^{+\infty} a_{m\tau}^s U_{m\tau}^0 e^{im\varphi}\frac{K_{i\tau}(qr)}{\sqrt{r}}d\tau, \tag{10}$$

$$a_{m\tau}^s = \frac{(-1)^m}{4}\frac{e^{-st_0}}{\cosh \pi\tau}e^{-im\varphi_0}\cdot\frac{\Gamma(1/2 - m + i\tau)}{\Gamma(1/2 + m + i\tau)}\cdot\frac{K_{i\tau}(qr)}{\sqrt{r}},$$

$$U_{m\tau}^0(\vartheta,\vartheta_0) = \begin{cases} P_{-1/2+i\tau}^m(\cos\vartheta)P_{-1/2}^m(-\cos\vartheta_0), \vartheta < \vartheta_0, \\ P_{-1/2+i\tau}^m(-\cos\vartheta)P_{-1/2}^m(\cos\vartheta_0), \vartheta_0 < \vartheta. \end{cases}$$

Let's express G_1^s via the Kontorovich-Lebedev integral (9), (10)

$$G_1^s = \frac{2}{\pi^2}\int_0^{+\infty} \tau \sinh \pi\tau \sum_{m=-\infty}^{+\infty} b_{m\tau}^s U_{m\tau}^1 \frac{K_{i\tau}(qr)}{\sqrt{r}}d\tau, \tag{11}$$

$$b_{m\tau}^s = -a_{m\tau}^s P_{-1/2+i\tau}^m(-\cos\vartheta_0)P_{-1/2+i\tau}^m(\cos\gamma), \; \gamma < \vartheta_0 < \pi,$$

$$U^1_{m\tau} = \sum_{n=-\infty}^{+\infty} x_{m,n+m_0}(\tau) \frac{P^{m+nN}_{-1/2+i\tau}(\pm\cos\vartheta)}{P^{m+nN}_{-1/2+i\tau}(\pm\cos\gamma)} e^{i(m+nN)\varphi},$$

$x_{m,n+m_0}$ are unknown coefficients, $P^m_\mu(\pm\cos\vartheta)$ are associated Legendre functions, the upper sign corresponds to $0<\vartheta<\gamma$ and the lower one to $\gamma<\vartheta<\pi$, $\Gamma(z)$ is a gamma-function, $m/N=\nu+m_0, -1/2\le\nu<1/2$, m_0 is the nearest integer to m/N. In order to obtain function equations for determining $x_{m,n}$ one should apply the boundary conditions and the continuity conditions for $\partial G^s_1/\partial\vartheta$ in slots; as a result we have

$$\sum_{n=-\infty}^{+\infty} x_{m,n} e^{inN\varphi} = e^{im_0\varphi}, \pi d/l < |N\varphi| \le \pi, \tag{12}$$

$$\sum_{n=-\infty}^{+\infty} N(n+\nu)\frac{|n|}{n}(1-\varepsilon_{m,n})x_{m,n} e^{iN\varphi} = 0, |N\varphi| < \pi d/l, \tag{13}$$

where

$$N(n+\nu)\frac{|n|}{n}(1-\varepsilon_{m,n}) = \frac{(-1)^{(n+\nu)N}}{\pi}\cdot\cosh\pi\tau\cdot\frac{\Gamma(1/2+i\tau+(n+\nu)N)}{\Gamma(1/2+i\tau-(n+\nu)N)}\cdot$$
$$\frac{1}{P^{(n+\nu)N}_{-1/2+i\tau}(\cos\gamma)P^m_{-1/2+i\tau}(-\cos\gamma)}, \tag{14}$$

$$\varepsilon_{m,n} = O\left(\frac{\sin^2\gamma}{N^2(n+\nu)^2}\right), N(n+\nu) >> 1.$$

Let's introduce

$$y_{m,n} = (-1)^{n-m_0}\frac{n+\nu}{m_0+\nu}\cdot\frac{|n|}{n}(1-\varepsilon_{m,n})x_{m,n},$$

$$1-\delta_{m,n} = \frac{1}{1-\varepsilon_{m,n}},\ \psi = N\varphi - \frac{|\varphi|}{\varphi},$$

and reduce (12), (13) to the following equations

$$\sum_{n=-\infty}^{+\infty}\frac{|n|}{n}(1-\delta_{m,n})y_{m,n}e^{in\psi} = e^{im_0\psi}, |\psi| < \pi(l-d)/l, \tag{15}$$

$$\sum_{n=-\infty}^{+\infty} y_{m,n}e^{in\psi} = 0, \qquad \pi(l-d)/l < |\psi| \le \pi, \tag{16}$$

with the complementary condition

$$\sum_{n=-\infty}^{+\infty}\frac{1}{n+\nu}\frac{|n|}{n}(1-\delta_{m,n})y_{m,n} = \frac{1}{m_0+\nu}. \tag{17}$$

By using the Riemann-Hilbert problem method [4,5] for a unit circle arc we bring the functional equations (15)-(17) to the system of linear algebraic equations for coefficients $y_{m,n}$

$$M_\nu(u) y_{m,0} = V^{m_0}(u) + \sum_{s=-\infty}^{+\infty} \frac{|s|}{s} \delta_{m,s} V^s(u) y_{m,s}, \tag{18}$$

$$M_\nu(u) = \frac{2}{\nu} \frac{P_\nu(-u) - P_\nu(-u)}{P_{\nu-1}(-u) + P_\nu(-u)}$$

$$y_{m,q} = V_{q-1}^{m_0}(u) + \sum_{s=-\infty}^{+\infty} \frac{|s|}{s} \delta_{m,s} y_{m,s} V_{q-1}^{s-1}(u) + y_{q,0} P_q(u), n \neq 0, \tag{19}$$

$$V_{n-1}^{m-1}(u) = \frac{n}{2(n-m)} [P_{n-1}(u) P_m(u) - P_n(u) P_{m-1}(u)]$$

$$V^p(u) = \frac{1}{p+\nu} \left\{ P_p(u) - \frac{P_{\nu-1}(u)}{P_\nu(-u) + P_{\nu-1}(-u)} \left[P_p(u) - P_{p-1}(u) \right] \right\}$$

$$u = \cos\left(\frac{l-d}{l}\pi\right)$$

The matrix operator of the system (18), (19) is compact and it's coefficients are independent on parameter q. For any problem parameters the system solution can be obtained with the reduction method and for a partly transmitted cone $(N >> 1, (l-d)/l << 1)$ with the iteration one too. The inversion of G^s is accomplished by procedure in [1]. It follows that

$$G_1 = \frac{c}{4\pi r r_0} \eta\left(\hat{t} - \frac{r + r_0}{c} \right) \sum_{m=-\infty}^{+\infty} e^{im\varphi} \int_0^{+\infty} g_{m\tau} U_{m\tau}^1 \cdot P_{-1/2+i\tau}(\cosh b) d\tau, \tag{20}$$

$$g_{m\tau} = (-1)^{m+1} e^{-im\varphi_0} \tau \tanh \pi\tau \cdot \frac{\Gamma(1/2 - m + i\tau)}{\Gamma(1/2 + m + i\tau)} P_{-1/2+i\tau}^m(-\cos\vartheta_0) P_{-1/2+i\tau}^m(\cos\gamma),$$

$$\cosh b(\hat{t}) = \frac{\hat{t}^2 c^2 - r^2 - r_0^2}{2 r r_0}, \ \hat{t} = t - t_0,$$

here $\eta(z)$ is the Heaviside unit function.

3. RESULTS

Let's consider a partly transmitted cone that is defined by existence of the limit

$$\lim_{\substack{N\to+\infty \\ d/l\to 1}}\left[-\frac{1}{N}\ln(1-d/l)\right]=Q>0.$$

Such a surface can be regarded as a model of a cone antenna formed by a great number of thin conductors. For the partly transmitted cone we obtain

$$G_1=\frac{c}{4\pi r r_0}\eta\left(\hat{t}-\frac{r+r_0}{c}\right)\sum_{m=-\infty}^{+\infty}e^{im\varphi}\int_0^{+\infty}h_{m\tau}\cdot P^m_{-1/2+i\tau}(\cos\vartheta)\cdot P_{-1/2+i\tau}(\cosh b)d\tau,\ 0<\vartheta<\gamma, \quad (21)$$

$$G_1=\frac{c}{4\pi r r_0}\eta\left(\hat{t}-\frac{r+r_0}{c}\right)\sum_{m=-\infty}^{+\infty}e^{im\varphi}\int_0^{+\infty}h_{m\tau}\frac{P^m_{-1/2+i\tau}(\cos\gamma)}{P^m_{-1/2+i\tau}(-\cos\gamma)}\cdot P^m_{-1/2+i\tau}(-\cos\vartheta)\cdot P_{-1/2+i\tau}(\cosh b)d\tau$$

$$\gamma<\vartheta<\pi, \quad (22)$$

$$h_{m\tau}=\frac{g_{m\tau}}{1+2mQ(1-\varepsilon_{m,0})}.$$

In order to simplify (21), (22) we assume first that the source point is on the cone axis $(\vartheta_0=\pi,\varphi_0=0;m=0)$. Then (21), (22) reduce to

$$G_1=\frac{-c}{4\pi r r_0}\eta\left(\hat{t}-\frac{r+r_0}{c}\right)\int_0^{+\infty}\frac{\tau th\pi\tau}{D_{i\tau}}P_{-1/2+i\tau}(\cos\gamma)P_{-1/2+i\tau}(-\cos\gamma)P_{-1/2+i\tau}(\cos\vartheta)P_{-1/2+i\tau}(\cosh b)d\tau$$

$$0<\vartheta<\gamma, \quad (23)$$

$$G_1=-\frac{c}{4\pi r r_0}\eta\left(\hat{t}-\frac{r+r_0}{c}\right)\int_0^{+\infty}\frac{\tau th\pi\tau}{D_{i\tau}}\left[P_{-1/2+i\tau}(\cos\gamma)\right]^2P_{-1/2+i\tau}(-\cos\vartheta)\cdot P_{-1/2+i\tau}(\cosh b)d\tau,$$

$$\gamma<\vartheta<\pi, \quad (24)$$

$$D_{i\tau}=\pi P_{-1/2+i\tau}(\cos\gamma)P_{-1/2+i\tau}(-\cos\gamma)+2Q\cosh\pi\tau.$$

Applying the residue theorem to the integrals in (23), (24) one may derive series representations for G_1. It follows from (23)

$$G_1=\frac{c}{2\pi r r_0}\sum_{j=0}^{+\infty}\frac{\mu_j}{\frac{d}{d\mu}D_{\mu_j}}P_{-1/2+\mu_j}(\cos\gamma)P_{-1/2+\mu_j}(-\cos\gamma)P_{-1/2+\mu_j}(\cos\vartheta)Q_{-1/2+\mu_j}(\cosh b),$$

$$c\hat{t}>r+r_0,\ 0<\vartheta<\gamma, \quad (26)$$

where $Q_\mu(z)$ is the Legendre function, μ_j are the positive roots of

$$D_\mu=0. \quad (27)$$

In the case of the partly transmitted cone the boundary problem spectrum is defined by the roots μ_j of (27).

For $Q \ll 1$

$$\mu_j^{\pm} = \alpha_j^{\pm} - \frac{2Q\cos\pi\alpha_j^{\pm}}{\pi \frac{d}{d\mu}\left[P_{-1/2+\mu}(\cos\gamma)P_{-1/2+\mu}(-\cos\gamma)\right]_{\mu=\alpha_j^{\pm}}} + O(Q^2), \tag{28}$$

$P_{-1/2+\alpha_j^{+}}(\cos\gamma) = 0$, $P_{-1/2+\alpha_j^{-}}(-\cos\gamma) = 0$;

for $Q \gg 1$

$$\mu_j = \frac{1}{2} + j + \frac{1}{2Q}\left[P_j(\cos\gamma)\right]^2 + O(Q^{-2}),\ j = 0,1,2,... \tag{29}$$

Taking into account the asymptoticbehaviour for the Legendre function $Q_\mu(z), |z| \gg 1$ [6] one may approximate (26) by the leading term in the series for the long-time response $(\hat{t} \gg 1)$

$$G_1 \sim \frac{c}{2\pi r r_0}\frac{\mu_0}{\frac{d}{d\mu}D_{\mu_0}} P_{-1/2+\mu_0}(\cos\gamma)P_{-1/2+\mu_0}(-\cos\gamma)P_{-1/2+\mu_0}(\cos\vartheta)Q_{-1/2+\mu_0}\left(\frac{\hat{t}^2c^2 - r^2 - r_0^2}{2rr_0}\right)$$

$$0 < \vartheta < \gamma, \tag{30}$$

where μ_0 is the smallest positive root of (27). For the special case of the partly transmitted cone $(Q \gg 1)$

$$\mu_0 = \frac{1}{2} + \frac{1}{2Q} + O(Q^{-2})$$

and

$$\frac{d}{d\mu}D_{\mu_0} = -2\pi Q\left[1 - \frac{1}{Q}\ln(0.5\sin\gamma) + O(Q^{-2})\right],\ \frac{1}{Q}\left|\ln(0.5\sin\gamma)\right| \ll 1.$$

Then we have

$$G_1 \sim -\frac{c}{4\pi^2 Q}(\hat{t}c)^{-2},\ \hat{t} \gg 1,\ 0 < \vartheta < \gamma. \tag{31}$$

It should be noted that the nonstationary boundary problem spectrum is the same as for the stationary boundary one [1].

4. CONCLUSIONS

Initial boundary value problem about constructing the time-dependent Green's function for a perfectly conducting cone with periodical longitudinal slots is considered. For a partly transmitted cone and the cone with small angles the solutions are expressed in simple closed forms. The boundary spectrum is investigated for special cases of the partly transmitted cone. It is shown that the nonstationary problem spectrum is the same as for the stationary one. The slot influence is studied.

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