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ELECTRODYNAMICS AND WAVE PROPAGATION

The Scattering of Plane Electromagnetic Waves from a Cone with Longitudinal Slots

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Abstract—The paper deals with the boundary-value problem of scattering of a plane electromagnetic wave from a semi-infinite ideally conducting cone with longitudinal slots cut at regular intervals. The problem is solved using the Kontorovich-Lebedev integral transformation and the semi-inverse method. In particular cases of a semitransparent cone, a single narrow conical ribbon, and a cone with narrow slot, an analytical solution is derived that is used to study the structure and polarization of the scattered field, as well as the field behavior in the vicinity of the cone vertex.

INTRODUCTION

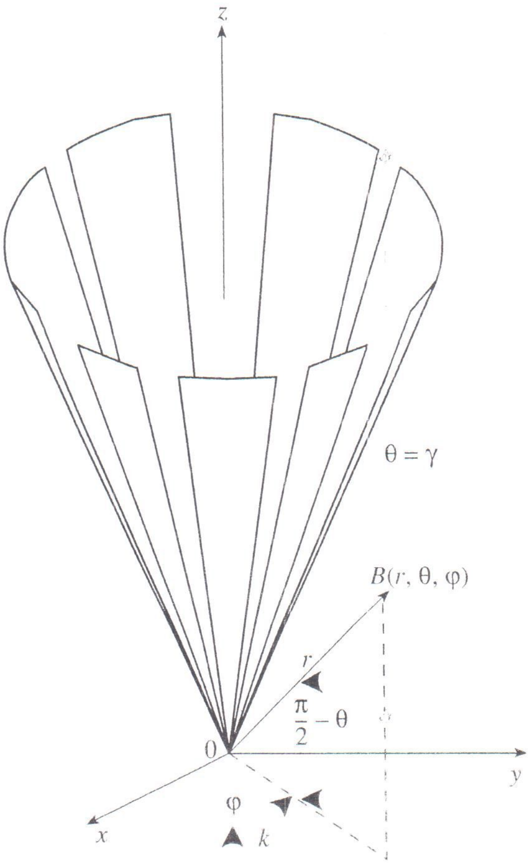
The results of investigation of the boundary-value problem on the electric-dipole excitation of a cone with longitudinal slots are given in [1]. However, of great interest from the standpoint of measurement and radar applications is the solution of the diffraction problem for such a structure. This paper, which is a continuation of [1], gives a solution to the problem of scattering of a plane electromagnetic wave from a semi-infinite ideally conducting cone with longitudinal slots cut at regular intervals.

1. FORMULATION OF THE PROBLEM AND METHOD OF SOLUTION

Let a homogeneous plane electromagnetic wave, which propagates along the axis of a thin ideally conducting semi-infinite circular cone with N slots cut at regular intervals along the generatrices, be incident on that cone (see the figure). The time dependence is given in the form $\exp(-i\omega t)$. In the employed (r, ϑ, φ)–spherical coordinate system, the conical surface is defined by the equation $\vartheta = \gamma$. The interval between slots in the structure being treated $l = 2\pi/N$ and the slot width d represent the magnitudes of dihedral angles formed by planes passed through the cone axis and the edges of adjacent conical ribbons. The presence of a cone with longitudinal slots leads to the emergence of a secondary field $\vec{E}^{(d)}$, $\vec{H}^{(d)}$. The total field for $\vec{E}$, $\vec{H}$ has the form

$$\vec{E} = \vec{E}^{(in)} + \vec{E}^{(d)}, \quad \vec{H} = \vec{H}^{(in)} + \vec{H}^{(d)},$$

where $\vec{E}^{(in)}$, $\vec{H}^{(in)}$ denote the plane wave field (primary field) which satisfies the set of Maxwell equations outside of the cone, the boundary conditions on the ribbons $\vec{E}_{tg}|_{\Sigma} = 0$, and the conditions of emission and finiteness of energy. These conditions provide for the uniqueness of the solution of the boundary-value diffraction problem set. We will express the components of the electromagnetic field in terms of the electric $V^{(1)}$ and magnetic $V^{(2)}$ Debye potentials and then reduce the initial electrodynamic problem to two scalar boundary-value Dirichlet and Neumann problems for the $V^{(1)}$ and $V^{(2)}$ potentials. The Debye potentials satisfy the homo-



The geometry of the structure.

geneous Helmholtz equation outside of the cone, the Dirichlet (for $V^{(1)}$) or Neumann (for $V^{(2)}$) boundary condition, the condition of emission, and the condition in the vicinity of boundary irregularities (cone vertex, ribbon edges). According to the total field structure,

$$V^{(s)} = V_{in}^{(s)} + V_d^{(s)}, \quad s = 1, 2.$$

We will use the Kontorovich-Lebedev integral transformation for solving scalar boundary-value problems [1],

$$G(\tau) = \int_0^{+\infty} g(r)\exp\left(-\frac{\pi\tau}{2}\right)\frac{H_{i\tau}^{(1)}(kr)}{\sqrt{r}}dr, \tag{1}$$

$$g(r) = -\frac{1}{2}\int_0^{+\infty} \tau\sinh\pi\tau\exp\left(-\frac{\pi\tau}{2}\right)G(r)\frac{H_{i\tau}^{(1)}(kr)}{\sqrt{r}}d\tau, \tag{2}$$

$H_{i\tau}^{(1)}(kr)$ is the Hankel function of the first kind, and $k = \omega/c$ is the wave number.

We will treat two cases of polarization of the primary field, namely,

$$\text{A. } \vec{E}^{in} = (E_x^{in}, 0, 0), \quad \vec{H}^{in} = (0, H_y^{in}, 0),$$

$$E_x^{in} = H_y^{in} = \exp(ikz)$$

and

$$\text{B. } \vec{E}^{in} = (0, E_y^{in}, 0), \quad \vec{H}^{in} = (H_x^{in}, 0, 0),$$

$$H_x^{in} = -E_y^{in} = \exp(ikz).$$

Case A

$$E_x^{in} = H_y^{in} = \exp(ikz).$$

Here, the Debye potentials corresponding to the plane wave field are determined, according to [2], in the form

$$V_{in}^{(1)} = -\frac{\cos\varphi}{k^2 r\sin\vartheta}$$

$$\times(\cos kr + i\cos\vartheta\sin kr - \exp(ikr\cos\vartheta)),$$

$$V_{in}^{(2)}(r, \vartheta, \varphi) = V_{in}^{(1)}\left(r, \vartheta, \frac{\pi}{2} - \varphi\right).$$

We will represent these potentials using the Kontorovich-Lebedev integrals,

$$V_{in}^{(1)} = \frac{1}{k}\sqrt{\frac{\pi}{2k}}\exp\left(i\frac{\pi}{4}\right)\cos\varphi\int_0^{+\infty}\tau\tanh\pi\tau\exp\left(-\frac{\pi\tau}{2}\right)$$

$$\times\frac{H_{i\tau}^{(1)}(kr)}{\sqrt{r}}P_{-1/2+i\tau}^{-1}(\cos\vartheta)d\tau + i\cos\varphi\tan\frac{\vartheta}{2}\frac{\sin kr}{k^2 r}, \tag{3}$$

where $P_{-1/2+i\tau}^{m}(\cos\vartheta)$ denotes associated Legendre functions. We will introduce the following notation:

$$V_{ind}^{(s)} = \sum_{m=-1;\,1}\exp(im\varphi)\int_0^{+\infty}\frac{H_{i\tau}^{(1)}(kr)}{\sqrt{r}}C_{m\tau}^{(s)}P_{-1/2+i\tau}^{m}(\cos\vartheta)d\tau, \tag{4}$$

$$V_{inc}^{(s)} = \frac{1}{2k^2}\frac{\sin kr}{r}\sum_{m=-1;\,1}\left(\frac{|m|}{m}\right)^{s-1}P_0^{-|m|}(\cos\vartheta)\exp(im\varphi), \tag{5}$$

$$s = 1, 2,$$

where

$$C_{m\tau}^{(s)} = \left(-i\frac{|m|}{m}\right)^{s-1}\sqrt{\frac{\pi}{2k}}\exp\left(i\frac{\pi}{4}\right)$$

$$\times\tau\tanh\pi\tau\exp\left(-\frac{\pi\tau}{2}\right)\left(\tau^2 + \frac{1}{4}\right)^{\alpha(m)}, \quad \alpha(m) = -\frac{1}{2}\left(\frac{|m|}{m} + 1\right).$$

Here,

$$V_{in}^{(s)} = V_{ind}^{(s)} + V_{inc}^{(s)}. \tag{6}$$

By analogy with (3)–(6), the Debye potentials $V_d^{(s)}$ corresponding to the secondary field will be represented as

$$V_d^{(s)} = V_{dd}^{(s)} + V_{dc}^{(s)}, \tag{7}$$

$$V_{dd}^{(s)} = -\int_0^{+\infty}\frac{H_{i\tau}^{(1)}(kr)}{\sqrt{r}}\sum_{m=-1;\,1}C_{m\tau}^{(s)}\frac{d^{s-1}}{d\gamma^{s-1}}P_{-1/2+i\tau}^{m}(\cos\gamma)U_{m\tau}^{(s)}d\tau, \tag{8}$$

$$U_{m\tau}^{(s)} = \sum_{n=-\infty}^{\infty}x_{m+nN}^{(s)}\frac{P_{-1/2+i\tau}^{m+nN}(\pm\cos\vartheta)}{\dfrac{d^{s-1}}{d\gamma^{s-1}}P_{-1/2+i\tau}^{m+nN}(\pm\cos\gamma)} \tag{9}$$

$$\times\exp(i(m+nN)\varphi),$$

$$V_{dc}^{(s)} = -\frac{i}{2k^2}\frac{\sin kr}{r}\frac{d^{s-1}}{d\gamma^{s-1}}\left(\tan\frac{\gamma}{2}\right)\sum_{m=-1;\,1}\left(-i\frac{|m|}{m}\right)^{s-1}U_m^{(s)}, \tag{10}$$

$$U_m^{(s)} = \sum_{n=-\infty}^{\infty}\xi_{m+nN}^{(s)}\frac{P_0^{-|m+nN|}(\pm\cos\vartheta)}{\dfrac{d^{s-1}}{d\gamma^{s-1}}P_0^{-|m+nN|}(\pm\cos\gamma)} \tag{11}$$

$$\times\exp(i(m+nN)\varphi),$$

where $x_p^{(s)}$ and $\xi_p^{(s)}$ are unknown coefficients related by

$$\xi_p^{(s)} = \lim_{i\tau \to 1/2} x_p^{(s)}. \tag{12}$$

The upper signs in (9) and (11) correspond to the $0 < \vartheta < \gamma$ range, and the lower signs correspond to the $\gamma < \vartheta < \pi$ range.

We use the boundary condition on the conical ribbons and the condition of continuity on the slots to derive the set of functional equations in $x_n^{(s)}$,

$$\sum_{n=-\infty}^{\infty} x_n^{(s)} \exp(inN\varphi) = \exp(im_0 N\varphi), \quad \frac{\pi d}{l} < |N\varphi| \le \pi, \tag{13}$$

$$\sum_{n=-\infty}^{\infty} x_n^{(s)} [N(n+\nu)]^{\chi(s)} \frac{|n|}{n}(1-\varepsilon_n^{(s)}) \exp(inN\varphi) = 0, \quad |N\varphi| < \frac{\pi d}{l}, \tag{14}$$

where

$$[N(n+\nu)]^{\chi(s)} \frac{|n|}{n}(1-\varepsilon_n^{(s)}) = \frac{(-1)^{(n+\nu)N+s-1}}{\pi \sin\gamma} \frac{\cosh\pi\tau\,\Gamma(1/2+i\tau+(n+\nu)N)}{\Gamma(1/2+i\tau-(n+i)N)}$$

$$\times \frac{1}{\dfrac{d^{s-1}}{d\gamma^{s-1}} P_{-1/2+i\tau}^{(n+\nu)N}(\cos\gamma) \dfrac{d^{s-1}}{d\gamma^{s-1}} p_{-1/2+i\tau}^{(n+\nu)N}(-\cos\gamma)}$$

$$(\chi(s) = (-1)^{s+1}),$$

$m/N = m_0 + \nu$, $-1/2 \le \nu < 1/2$, m_0 is an integer closest to m/N, and $\Gamma(z)$ is the gamma function.

The quantity $\varepsilon_n^{(s)}$ is estimated at $N(n+\nu) \gg 1$,

$$\varepsilon_n = O\left(\frac{1}{N^2(n+\nu)^2}\right). \tag{15}$$

When the method of the Riemann–Hilbert problem is used [1], the functional equations (13) and (14) reduce to two independent infinite systems of Fredholm-type linear algebraic equations (ISLAEs) of the second kind in $x_n^{(s)}$. Following are the ISLAEs for $x_n^{(2)}$,

$$B_\nu(u) x_0^{(2)} = -\frac{|m_0|}{m}(1-\varepsilon_{m_0}^{(2)}) V^{m_0}(u) + \sum_{p=-\infty}^{\infty} (x_p^{(2)} - \delta_p^{m_0}) \frac{|p|}{p} \varepsilon_p V^p(u), \tag{16}$$

$$B_\nu(u) = \frac{1}{\nu} \frac{2P_{\nu-1}(-u)}{P_{\nu-1}(-u) + P_\nu(-u)},$$

$$x_0^{(2)} - \delta_0^{m_0} = D_{m_0} V_{n-1}^{m_0-1}(u) + \sum_{p \ne 0} (x_p^{(2)} - \delta_p^{m_0}) \frac{|p|}{p} \varepsilon_p^{(2)} \times V_{n-1}^{p-1}(u) + x_0^{(2)} [P_n(u) + \varepsilon_0^{(2)} V_{n-1}^{-1}(u)], \quad n \ne 0, \tag{17}$$

where

$$D_{m_0} = -\frac{|m_0|}{m}(1-\varepsilon_{m_0}^{(2)}), \quad u = \cos\frac{\pi d}{l},$$

$$\delta_p^m = \begin{cases} 0, & m \ne p, \\ 1, & m = p, \end{cases}$$

and $V_{n-1}^{p-1}(u)$, $V^p(u)$, and $V_{n-1}^{-1}(u)$ are known functions [1].

It is not difficult to demonstrate that the matrix operators of two ISLAEs are compact in the Hilbert space $\tilde{l}_2^{(s)}$ of the $\{x_n^{(s)}\}_{n=-\infty}^{+\infty}$ sequences with the scalar product

$$x^{(s)} \cdot y^{(s)} = \sum_{n=-\infty}^{\infty} (1+|n|)^{\chi(s)} x_n^{(s)} y_n^{(s)}.$$

These systems have a unique solution which, for any correlations between ribbon width and the period for arbitrary finite values of parameters N, m, and τ may be derived by the reduction method. Note that the coefficients $x_n^{(s)}$ do not depend on the wave number, and this is convenient for constructing directional patterns and for studying the behavior of the field in the vicinity of the cone point ($kr \ll 1$). In particular cases of conical structure, when the slots are numerous, and their width is either small or comparable with the interval (semitransparent cone) between narrow slots or narrow conical ribbons, the norms of system operators are less than unity, which enables one to use the method of successive approximations for solving the systems.

Case B

Here, the Debye potentials $\tilde{V}_{in}^{(s)}$ corresponding to the primary field are defined by the relations

$$\tilde{V}_{in}^{(1)}(r, \vartheta, \varphi) = -V_{in}^{(1)}\left(r, \vartheta, \frac{\pi}{2} - \varphi\right),$$

$$\tilde{V}_{in}^{(2)}(r, \vartheta, \varphi) = V_{in}^{(1)}(r, \vartheta, \varphi),$$

where the tilde indicates the Debye potential for case B.

The structures of the potentials $\tilde{V}_d^{(s)}$ and their representation in the form of the Kontorovich-Lebedev integral are analogous to (7)–(12), as in case A. The unknown Fourier coefficients in the expansions of potentials $\tilde{V}_d^{(s)}$ in series with respect to angle φ satisfy the same functional equations and two ISLAEs as the coefficients $x_n^{(s)}$ in (13), (14), (16), and (17).

2. ANALYSIS OF ANALYTICAL SOLUTION

In the cases of a semitransparent cone, of a single narrow ribbon, and of a cone with a single slot, the solution of two ISLAEs is found by the method of successive approximations to derive an analytical solution of the boundary-value problem for two different conditions.

Case A

Semitransparent cone. We will treat two cases of a semitransparent cone, which are defined by the existence of limits

$$\text{(a)}\ \lim_{\substack{N\to\infty\\ d/l\to 1}}\left[-\frac{1}{N}\ln\cos\frac{\pi d}{2l}\right] = Q, \tag{18}$$

$$\text{(b)}\ \lim_{\substack{N\to\infty\\ d/l\to 0}}\left[-\frac{1}{N}\ln\sin\frac{\pi d}{2l}\right] = W. \tag{19}$$

(a) At $Q \neq 0$ and $W = 0$, $V_d^{(2)} = 0$, and the secondary field is described by the potential $V_d^{(1)}$ which has the form

$$V_d^{(1)} = -\frac{1}{k}\cos\varphi\int_0^{+\infty}\frac{H_{i\tau}^{(1)}(kr)}{\sqrt{r}}\frac{(\tau^2+1/4)^2}{\sigma_{i\tau}}C_{i\tau}^{(1)}$$
$$\times[P_{-1/2+i\tau}^{-1}(\cos\gamma)]^2P_{-1/2+i\tau}^{-1}(-\cos\vartheta)d\tau$$
$$-\frac{i}{k^2}\frac{\cos\varphi}{1+2Q}\tan^2\frac{\gamma}{2}\cot\frac{\vartheta}{2}\frac{\sin kr}{r},\quad \gamma<\vartheta<\pi, \tag{20}$$
$$\sigma_{i\tau} = (\tau^2+1/4)P_{-1/2+i\tau}^{-1}(\cos\gamma)$$
$$\times P_{-1/2+i\tau}^{-1}(-\cos\gamma)+2Q\frac{\cosh\pi\tau}{\pi}.$$

An analogous representation takes place at $0<\vartheta<\gamma$. The components of the electromagnetic field on the cone surface satisfy the averaged boundary conditions

$$E_r^+ = E_r^-,$$

$$\frac{ik}{\arcsin\gamma}E_r = \left(\frac{\partial^2}{\partial r^2}+k^2\right)(r\hat{H}\varphi),$$

$$\hat{H} = H^+-H^-,\quad f^\pm = f|_{\vartheta=\gamma\pm 0},$$

which coincide, within the sign of the wave number, with the averaged boundary conditions in the case of the excitation of a semitransparent cone of the same type by a radial electric dipole [1]. We proceed in (20) to integration with respect to imaginary axis and represent $V^{(1)}$ by a series of residues over the integrand poles

$$V^{(1)} = -\frac{\pi i}{k}\sqrt{\frac{2\pi}{kr}}\exp\left(i\frac{\pi}{4}\right)\cos\varphi\sum_{q=1}^{+\infty}\frac{\mu(\mu^2-1/4)I_\mu(kr)}{\cos\pi\mu\frac{d}{d\mu}\sigma_\mu} \tag{21}$$
$$\times[P_{-1/2+\mu}^{-1}(\cos\gamma)]^2P_{-1/2+\mu}^{-1}(-\cos\vartheta)|_{\mu=\tilde{\nu}_q},$$

$$\sigma_{\tilde{\nu}_q} = 0. \tag{22}$$

Note that series (21) diverges rapidly when the observation point is in the vicinity of the vertex ($kr \ll 1$); therefore, it may be used conveniently to clarify the field behavior in the vicinity of the cone point. At $kr \gg 1$, the series converges slowly; for analyzing the field in the far zone, it is expedient to use the integral representation (20).

In what follows, by the spectrum of eigenvalues of the boundary-value problem is meant the set of integrand poles in the integral representation for Debye potentials corresponding to the total field. The spectrum of the problem for semitransparent cone (18) is defined by the roots of Eq. (22), of which the least characterizes the field behavior at the vertex. In some particular cases, we will use the asymptotics for $\tilde{\nu}_q$,

(i) $Q \ll 1$,

$$\tilde{\nu}_q = \alpha_q^{1\pm}(\gamma)$$
$$+\left.\frac{2Q\cos\pi\mu}{\pi\left(\mu^2-\frac{1}{4}\right)\frac{d}{d\mu}\left[P_{-\frac{1}{2}+\mu}^{-1}(\cos\gamma)P_{-\frac{1}{2}+\mu}^{-1}(-\cos\gamma)\right]}\right|_{\mu=\alpha_q^{1\pm}}$$
$$+O(Q^2),$$

$$P_{-\frac{1}{2}+\alpha_q^{1+}}^{-1}(\cos\gamma) = 0,\quad P_{-\frac{1}{2}+\alpha_q^{1-}}^{-1}(-\cos\gamma) = 0;$$

(ii) $Q \gg 1$,

$$\tilde{\nu}_q = \frac{1}{2}+q+\frac{1}{2Q}q(q+1)[P_q^{-1}(\cos\gamma)]^2+O(Q^{-2}),$$
$$q = 1, 2\ldots;$$

(iii) $\gamma \ll 1$,

$$\tilde{\nu}_q = \frac{1}{2}+q+\frac{q(q+1)}{4(1+2Q)}\gamma^2+O\left(\gamma^4\ln\frac{2}{\gamma}\right),\quad q = 1, 2\ldots$$

The components of the total electric field in the vicinity of the cone point behave as $(kr)^{-\frac{3}{2}+\tilde{\nu}}$, where $\tilde{\nu} = \min\tilde{\nu}_q$. The least root in cases (i)–(iii) is the ν_1 root. In view of the fact that, at $\gamma \ll 1$, the total electric field at the vertex of a solid cone behaves as $(kr)^{\gamma^2/2}$, we conclude that, in the case of the semitransparent cone ($\gamma \ll 1$) being treated, the total electric field decreases more slowly as the cone point is approached than in the case of the solid cone. The secondary field is defined by the electric Debye potential alone ($H_r^d = 0$), as a result of which this field is a field of the electric type (transverse-magnetic field). The plane wave field is defined by two potentials (4)–(6); therefore, it does not belong to either the electric or magnetic type.

(b) At $W \neq 0$ and $Q = 0$, the secondary field components are expressed in terms of both Debye potentials; in so doing, the electric potential is the same as in the case of a solid cone and experiencing no effect of the slots, and the magnetic potential is defined by the expression

$$V_d^{(2)} = 2\omega\frac{i}{k}\sin\varphi\int\limits_0^{+\infty}\frac{H_{i\tau}^{(1)}(kr)}{\sqrt{r}}\frac{C_{-1\tau}^{(2)}}{\Delta_{i\tau}}\frac{\frac{d}{d\gamma}P_{-\frac{1}{2}+i\tau}(\cos\gamma)}{\frac{d}{d\gamma}P^{-1}_{-\frac{1}{2}+i\tau}(-\cos\gamma)}$$

$$\times P^{-1}_{-\frac{1}{2}+i\tau}(-\cos\vartheta)d\tau + \frac{i}{k^2}\sin\varphi\frac{2W}{1+2W}\tan^2\frac{\gamma}{2}\cot\frac{\vartheta}{2}\frac{\sin kr}{r},$$

$$\gamma < \vartheta < \pi, \tag{23}$$

$$\Delta_{i\tau} = 2W$$

$$-\frac{\cosh\pi\tau}{\pi\sin\gamma}\frac{1}{\left(\tau^2+\frac{1}{4}\right)\frac{d}{d\gamma}P^{-1}_{-\frac{1}{2}+i\tau}(\cos\gamma)\frac{d}{d\gamma}P^{-1}_{-\frac{1}{2}+i\tau}(-\cos\gamma)}.$$

An analogous representation takes place at $0 < \vartheta < \gamma$. We pass on to the limit in (23) at $W \longrightarrow \infty$ (semitransparent cone transforms to solid) to derive the expression for the magnetic Debye potential as applied to a solid cone. The field components on the surface of a semitransparent cone satisfy the following averaged boundary conditions:

$$E_\varphi^+ = E_\varphi^-,$$

$$-\frac{ik}{4W\sin\gamma}\tilde{H}_r = \left(\frac{\partial^2}{\partial r^2}+k^2\right)(rE_\varphi).$$

In order to analyze the field behavior in the vicinity of the cone point, we will represent $V^{(s)}$ as a series with respect to integrand poles,

$$V^{(1)} = \frac{\pi i}{k}\sqrt{\frac{2\pi}{kr}}\exp\left(i\frac{\pi}{4}\right)\cos\varphi\sum_{n=1}^{+\infty}\frac{\mu}{\cos\pi\mu}I_\mu(kr)$$

$$\times\left.\frac{P^{-1}_{-\frac{1}{2}+\mu}(\cos\gamma)}{\frac{d}{d\mu}P^{-1}_{-\frac{1}{2}+\mu}(-\cos\gamma)}P^{-1}_{-\frac{1}{2}+\mu}(-\cos\vartheta)\right|_{\mu=\alpha_n^{1-}},$$

$$V^{(2)} = 2W\frac{\pi i}{k}\sqrt{\frac{2\pi}{kr}}\exp\left(i\frac{\pi}{4}\right)\sin\varphi\sum_{n=1}^{+\infty}\frac{\mu}{\cos\pi\mu}\frac{I_\mu(kr)}{\frac{d}{d\mu}\Delta_\mu} \tag{24}$$

$$\times\left.\frac{\frac{d}{d\gamma}P^{-1}_{-\frac{1}{2}+\mu}(\cos\gamma)}{\frac{d}{d\gamma}P^{-1}_{-\frac{1}{2}+\mu}(-\cos\gamma)}P^{-1}_{-\frac{1}{2}+\mu}(-\cos\vartheta)\right|_{\mu=\tilde{\nu}_n},$$

$$\gamma < \vartheta < \pi, \quad \Delta_{\tilde{\nu}_m} = 0.$$

We will investigate for some cases the roots of Eq. (24) that enter the spectrum of the boundary-value problem and define the field behavior in the vicinity of the cone point,

(i) $W \ll 1$,

$$\tilde{\nu}_n = \frac{1}{2}+n+2W\sin^2\gamma(-1)^n n(n+1)\frac{d}{d\gamma}P_n^{-1}(\cos\gamma)$$

$$\times\frac{d}{d\gamma}P_n^{-1}(-\cos\gamma)+O(W^2), \quad n = 1, 2\ldots;$$

(ii) $W\sin^2\gamma \gg 1$,

$$\tilde{\nu}_n = \varsigma_n^{1\pm}(\gamma) - \left.\frac{1}{2W\sin^2\gamma}\frac{\cos\pi\mu}{\pi\left(\mu^2-\frac{1}{4}\right)\frac{d}{d\mu}\left[\frac{d}{d\gamma}P^{-1}_{-\frac{1}{2}+\mu}(\cos\gamma)\frac{d}{d\gamma}P^{-1}_{-\frac{1}{2}+\mu}(-\cos\gamma)\right]}\right|_{\mu=\varsigma_n^{1\pm}} + O((W\sin^2\gamma)^{-1}),$$

$$\left.\frac{d}{d\gamma}P^{-1}_{-\frac{1}{2}+\mu}(\pm\cos\gamma)\right|_{\mu=\varsigma_n^{1\pm}} = 0;$$

(iii) $\gamma \ll 1$,

$$\hat{\nu}_n = \frac{1}{2} + n - n(n+1)\frac{W}{1+2W}\frac{\gamma^2}{2} + O\left(\gamma^4 \ln\frac{2}{\gamma}\right),$$

$$n = 1, 2\ldots$$

In (i)–(iii), the magnetic field in the vicinity of the cone point behaves as $kr^{\hat{\nu}}$, where

$$\hat{\nu}_n = \begin{cases} -\frac{1}{4}W\sin^2 2\gamma, & W \ll 1, \\ -\frac{3}{2} + \hat{\nu}_1, & W\sin^2\gamma \gg 1, \\ -\frac{W}{1+2W}\gamma^2, & \gamma \ll 1. \end{cases}$$

In view of the fact that the magnetic field at the vertex of a solid cone at $\gamma \ll 1$ has a singularity of the order of $(kr)^{\gamma^2/2}$, one can conclude that, in the case of a semi-transparent cone (19) and $\gamma \ll 1$, the singularity of the magnetic field in the vicinity of the vertex is weaker than in the case of a solid cone. The behavior of the electric field ($\gamma \ll 1$) at the vertex is asymptotically the same as in the case of a solid cone.

Narrow conical ribbon ($N = 1$, $\beta = 2\pi - d \ll 1$). We will designate the angular width of a conical ribbon as β. Away from the ribbon edges, the asymptotic expansion of electric potential with respect to the smallness parameter β has the form

$$V_d^{(1)} = -\frac{1}{2\ln\sin(\beta/4)}\frac{1}{k}\sum_{n=-\infty}^{+\infty}(-1)^n \exp(in\varphi)$$

$$\times \int_0^{+\infty} C_{-1\tau}^{(1)} P^{-1}_{-\frac{1}{2}+i\tau}(\cos\gamma)\frac{\frac{1}{|n|}(1-\delta_n^{(1)})}{M_{i\tau}} \tag{25}$$

$$\times \frac{P^n_{-\frac{1}{2}+i\tau}(\pm\cos\vartheta)}{P^n_{-\frac{1}{2}+i\tau}(\pm\cos\gamma)}\frac{H^{(1)}_{i\tau}(kr)}{\sqrt{r}}d\tau + \frac{i}{k^2}\frac{\sin kr}{r} + O\left(\frac{\beta^2}{\ln\frac{1}{\beta}}\right),$$

where $1 - \delta_n^{(1)} = \frac{1}{1-\varepsilon_n^{(1)}}$, $A_{i\tau} = \frac{\pi}{\cosh\tau}P_{-\frac{1}{2}+i\tau}(\cos\gamma) \times P_{-\frac{1}{2}+i\tau}(-\cos\gamma)$, and

$$M_{i\tau} = 1 - \frac{1}{2\ln\sin\frac{\beta}{4}}\left[A_{i\tau} - \sum_{p\neq 0}\frac{1}{|p|}\delta_p^{(1)}\right].$$

In this case, the magnetic Debye potential $V_d^{(2)}$ is of the order of $O\left(\beta^2/\ln\left(\frac{1}{\beta}\right)\right)$. Therefore, the dominant terms in the expansion of field components with respect to parameter β correspond to the dominant term in the expansion of electric Debye potential (25). The spectrum of the boundary-value problem is defined in this case by the roots of the equations

$$\frac{\cos\pi\mu}{\pi P_{-\frac{1}{2}+\mu}(\cos\gamma)P_{-\frac{1}{2}+\mu}(-\cos\gamma) - \cos\pi\mu\sum_{p\neq 0}\frac{1}{|p|}\delta_p^{(1)}} = \frac{1}{2\ln\sin\frac{\beta}{4}}, \tag{26}$$

$$\frac{\cos\pi\mu}{\pi\sin^2\gamma\left(\mu^2-\frac{1}{4}\right)P^{-1}_{-\frac{1}{2}+\mu}(\cos\gamma)P^{-1}_{-\frac{1}{2}+\mu}(-\cos\gamma)} = -\sin^2\frac{\beta}{4}, \tag{27}$$

$$\frac{\cos\pi\mu}{n\cos\pi\mu + \pi(-1)^{n+s}n^{\chi(s)+1}D_\mu^{(s)}} = -\sin^2\frac{\beta}{4},$$

$$D_\mu^{(s)} = (\sin\gamma)^{1-\chi(s)}\frac{\Gamma\left(\frac{1}{2}+\mu+n\right)}{\Gamma\left(\frac{1}{2}+\mu-n\right)}\frac{d^{s-1}}{d\gamma^{s-1}} \tag{28}$$

$$\times P^{-n}_{-\frac{1}{2}+\mu}(\cos\gamma)\frac{d^{s-1}}{d\gamma^{s-1}}P^{-n}_{-\frac{1}{2}+\mu}(-\cos\gamma).$$

The roots of Eqs. (27) and (28) are in the vicinity of zeros of $\cos\pi\mu$, with the exception of $\mu = 1/2$. The roots of (26) have the form

$$\varsigma_q = \frac{1}{2} + q - \frac{1}{2\ln\sin\frac{\beta}{4}}\left\{[P_q(\cos\gamma)]^2 + 2\sum_{p=1}^{q}\frac{(q+p)!}{(q-p)!}\right. \tag{29}$$

$$\left.\times[P_q^{-q}(\cos\gamma)]^2\right\} + O(\ln^{-2}\beta), \quad q = 0, 1, 2\ldots$$

The minimal eigenvalue of the spectrum is the least root of (29),

$$\varsigma_0 = \frac{1}{2} - \frac{1}{2\ln\sin\frac{\beta}{4}} + O\left(\ln^{-2}\left(\sin\frac{\beta}{4}\right)\right),$$

which characterizes the electric field behavior at the vertex of the conical ribbon. For example, component E_θ in the vicinity of the vertex behaves in the following manner:

$$E_\theta \approx -\frac{1}{2\ln\sin\frac{\beta}{4}}(kr)^{-\frac{3}{2}+\varsigma_0}F(\theta, \varphi),$$

where

$$F(\vartheta, \varphi) = \hat{A}\left\{\cot\frac{\vartheta}{2} + \frac{1}{\sin\vartheta}\left[-1\right.\right.$$

$$\left.\left. + \mathrm{Re}\left(\frac{1 - \hat{b}e^{i\varphi}}{\sqrt{\hat{b}^2\exp(2i\varphi) + 2\hat{b}\exp(i\varphi)\cos\frac{\beta}{2} + 1}}\right)\right]\right\}, \quad (30)$$

$$\gamma < \vartheta < \pi,$$

where $\hat{A}$ is the known coefficient, and $\hat{b} = \cot\frac{\vartheta}{2} \cdot \tan\frac{\gamma}{2}$.

For $0 < \vartheta < \gamma$, ϑ in (29) must be replaced by $\pi - \vartheta$.

Cone with narrow slot ($N = 1$, $d \ll 1$). It is known that the spectrum of the boundary-value problem for a solid cone, on which a plane wave is incident along its axis, consists of zeros of the functions $P^{-1}_{-\frac{1}{2}+\mu}(-\cos\gamma)$ and $\frac{d}{d\gamma}P^{-1}_{-\frac{1}{2}+\mu}(-\cos\gamma)$. In so doing, the least of the roots of these functions characterizes the behavior of the electric and magnetic fields at the cone vertex, respectively. The presence of a narrow slot perturbs the spectrum for a solid cone and causes a variation of the secondary field structure. Following are the expressions for eigenvalues in the case of a cone with a narrow slot,

$$\nu_n^{p\pm} = \alpha_n^{p\pm}(\gamma) - g_n^{p\pm}\sin^2\frac{d}{4} + O(d^4),$$

$$g_n^{p\pm} = \frac{\Gamma(1/2 + \mu - p)}{\Gamma(1/2 + \mu + p)}$$

$$\times \left.\frac{(-1)^p\cos\pi\mu}{\pi\frac{d}{d\mu}\left[P^{-p}_{-\frac{1}{2}+\mu}(\cos\gamma)P^{-p}_{-\frac{1}{2}+\mu}(-\cos\gamma)\right]}\right|_{\mu = \alpha_n^{p\pm}},$$

$$P^{-p}_{-\frac{1}{2}+\alpha_n^{p\pm}}(\pm\cos\gamma) = 0, \quad n, p = 0, 1, 2, \ldots;$$

$$\nu_n^{q\pm} = \varsigma_n^{q\pm}(\gamma) - b_n^{q\pm}\sin^2\frac{d}{4} + O(d^4),$$

$$b_n^{p\pm} = \frac{\Gamma\left(\frac{1}{2} + \mu - q\right)}{\Gamma\left(\frac{1}{2} + \mu + q\right)}$$

$$\times \left.\frac{(-1)^q\cos\pi\mu}{\pi\sin^2\gamma\frac{d}{d\mu}\left[\frac{d}{d\gamma}P^{-q}_{-\frac{1}{2}+\mu}(\cos\gamma)\frac{d}{d\gamma}P^{-q}_{-\frac{1}{2}+\mu}(-\cos\gamma)\right]}\right|_{\mu = \varsigma_n^{q\pm}},$$

$$\frac{d}{d\gamma}P^{-q}_{-\frac{1}{2}+\varsigma_n^{p\pm}}(\pm\cos\gamma) = 0, \quad q = 1, 2, \ldots;$$

$$\xi = \frac{1}{2} - \frac{1}{2\sin^2\gamma\ln\sin\frac{d}{4}} + O(\ln^{-2}d). \quad (31)$$

The electric field in the vicinity of the point of a cone with a narrow slot behaves as

$$(kr)^{-\frac{3}{2}+\nu_0^-}, \quad \gamma \le \pi/2,$$

where

$$\nu_0^- = \alpha_0^- - h_0^-\sin^2\frac{d}{4} + O(d^4),$$

$$\alpha_0^- = \alpha_0^{p-}\big|_{p=0}, \quad 0 < \alpha_0^-(\gamma) \le 3/2,$$

$$h_0^- = \left.\frac{\cos\pi\alpha_0^-}{\pi P_{-\frac{1}{2}+\alpha_0^-}(\cos\gamma)\frac{d}{d\mu}P_{-\frac{1}{2}+\mu}(-\cos\gamma)}\right|_{\mu = \alpha_0^-} > 0.$$

The values of $\alpha_0^-(\gamma)$ are given in [3]. In view of the fact that the electric field has no singularity at the vertex of a solid cone, because it is defined by the term of the order of $(kr)^{-\frac{3}{2}+\alpha_0^{1-}}$, $3/2 < \alpha_0^{1-}(\gamma)$, one can conclude that the presence of a narrow slot results in the emergence of a singularity of the electric field. As the cone point is approached, the magnetic field increases as $(kr)^{-\frac{3}{2}+\varsigma_0^{1-}}$ (the values of ς_0^{1-} are given in [3]). The components of the magnetic field in the vicinity of the vertex of a cone with a narrow slot have a singularity of the order of $(kr)^{-\frac{3}{2}+\xi}$. This is indicative of an increase in the singularity of the magnetic field (as compared with a solid cone).

The structure of a scattered field includes terms corresponding to the field for a solid cone and terms due to the presence of a slot. One of the latter terms is the wave corresponding to the eigenvalue of ξ (31). In addition to the roots of $\frac{d}{d\gamma}P^{-n}_{-\frac{1}{2}+\mu}(\pm\cos\gamma) = 0$, $n \ge 1$, the value of $\mu = 1/2$ that corresponds to a function making no contribution to the field is present in the spectrum of eigenvalues of the boundary-value problem of Neumann for magnetic potential in the case of a solid cone. In the presence of a narrow slot, the value of $\mu = 1/2$ is perturbed by the slot, and the function corresponding to this value introduces into the field a nonzero contribution and characterizes the singularity of the magnetic field in the vicinity of the cone point. When a plane electromagnetic wave is incident on a cone with narrow slot, a wave of the slot type is present in the secondary field structure, this latter wave corresponding to the value of ξ and defining the magnetic field behavior in the vicinity of the cone point.

Case B

Semitransparent cone

(I) $Q \neq 0$, $W = 0$:

$$\tilde{V}_d^{(1)}(r, \vartheta, \varphi) = -V_d^{(1)}\left(r, \vartheta, \frac{\pi}{2} - \varphi\right),$$

$$\tilde{V}_d^{(2)} = 0,$$

(II) $Q = 0$, $W \neq 0$.

The potential $V_d^{(1)}$ is the same as in the case of a solid cone, and

$$\tilde{V}_d^{(2)}(r, \vartheta, \varphi) = V_d^{(2)}\left(r, \vartheta, \frac{\pi}{2} - \varphi\right).$$

The spectrum of the boundary-value problem and the field behavior in the vicinity of the cone vertex are the same as in the case *A* of plane wave incidence.

Narrow conical ribbon. Following is the asymptotic expression for $\tilde{V}_d^{(1)}$ away from the ribbon edges,

$$\tilde{V}_d^{(1)} = \sin^2\frac{\beta}{2}\left\{-\frac{i}{k}\sum_{n\neq 0}(-1)^n\frac{|n|}{n}\exp(in\varphi)\int_0^{+\infty}\frac{C_{-1\tau}^{(1)}}{T_{i\tau}}\right.$$

$$\times P^{-1}_{-\frac{1}{2}+i\tau}(\cos\gamma)\frac{P^{n}_{-\frac{1}{2}+i\tau}(-\cos\vartheta)}{P^{n}_{-\frac{1}{2}+i\tau}(-\cos\gamma)}\frac{H_{i\tau}^{(1)}(kr)}{\sqrt{r}}d\tau + \frac{2i}{k^2}$$

$$\left.\times\frac{\tan^2\frac{\gamma}{2}\cot\frac{\vartheta}{2}\sin\varphi}{\tilde{b}^2 + 2\tilde{b}\cos\varphi + 1}\frac{\sin kr}{r}\right\} + O\left(\frac{\beta^2}{\ln\frac{1}{\beta}}\right), \quad \gamma < \vartheta < \pi,$$

where $T_{i\tau} = 1 - \delta_n^{(1)}\sin^2(\beta/2)$.

For a narrow ribbon, the spectrum is defined by the set of roots of Eqs. (26)–(28). The field behavior in the vicinity of the vertex does not depend on the polarization of the incident field and, therefore, its singularity at $kr \ll 1$ is the same as in case A (the difference resides in the amplitude factor containing the smallness parameter β).

CONCLUSION

The Kontorovich-Lebedev integral transformation was used in combination with the semi-inverse method to perform an analytic investigation of the boundary-value problem of scattering of a plane electromagnetic wave from an ideally conducting semi-infinite circular cone with slots cut at regular intervals. In the cases when the vector of the intensity of the electric or magnetic field of the plane wave propagating along the cone axis was directed on the abscissa of Cartesian coordinates, the problem was solved for a semitransparent cone, for a single narrow ribbon, and for a cone with a narrow slot. The spectrum of eigenvalues of the Dirichlet and Neumann boundary-value problems for the electric and magnetic Debye potentials was found. It has been demonstrated that the spectrum in the case of a cone with narrow slot is the spectrum (perturbed by the slot) of eigenvalues of the respective boundary-value problems for a solid cone. The polarization and structure of the scattered field were investigated in the treated cases of conical surface. In the cases involving a semitransparent cone, its polarization depends on the cone filling parameters Q and W.

An analysis of the solution for a cone with a narrow slot has demonstrated the presence in the scattered field structure of a slot-type wave whose properties are similar with those of a slot wave in a cylindrical slot line. The field behavior in the neighborhood of the boundary irregularities has been determined. In so doing, the field components perpendicular to the ribbon edge have a known root singularity, and the presence of a narrow slot causes an increase in the field singularity in the vicinity of the cone point as compared with the case of a solid cone. The employed approach to the solution of such a problem may be applied to the investigation of structures of a more complex conical geometry.

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