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PHYSICS

Meler–Fock Transformations in Problems of Wave Diffraction on Unclosed Structures in the Time Region

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INTRODUCTION

Based on concepts developed in [1, 2], we have applied for the first time the Kontorovich–Lebedev integral transformation to solve the first and second boundary value problems for Helmholtz equations with the three-dimensional unclosed biconic geometry. This has allowed us to lower the dimension of the equations and to obtain the solution of the electrodynamic problem in the frequency region [3]. In the present paper, we propose and substantiate a novel method for solving boundary value problems for the wave equation in wedge-like and conic regions. The method is based on the use of the Meler–Fock transformation [4] combined with the method of singular integral equations of pair summator equations [5]. The employment of this transformation to solving boundary value electrodynamic problems in the time region for unclosed conic structures makes it possible to find an analytical solution and to perform the qualitative analysis of their scattering properties.

FORMULATION OF THE PROBLEM

We consider the problem of wave diffraction on an unbounded thin conic structure Σ consisting of two cones Σ_1 and Σ_2 $\left(\Sigma = \bigcup_{j=1}^{2} \Sigma_j\right)$ with a common vertex and axis, opening angles $2\gamma_j$, and with N slots periodically cut along the generatrices and having the angular width d_j ($j = 1, 2$), respectively (Fig. 1). The structure period is $l = \frac{2\pi}{N}$, and d_j are the values of dihedral angles formed by the intersection of planes that contain the cone axis and slot edges. There is also a point field source located at the point $M_0(\mathbf{r}_0)$. The field generated by the source varies in accordance with a law given by the function $f(t - t_0)$ (the source is switched on at the time instant t_0). We now introduce the spherical coordinate system r, θ, φ with the origin at the vertex of the conic structure. In this system, each of the cones is determined by the equation Σ_j: $\theta = \gamma_j$. It is necessary to determine the potential $\upsilon^{(\chi)}(\mathbf{r}, t)$ that satisfies at every instant of time the following conditions:

the wave equation

$$\left(\Delta - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\right)\upsilon^{(\chi)}(\mathbf{r}, t) = -\tilde{F}^{(\chi)}(\mathbf{r}, t),$$
$$\mathbf{r} \notin \Sigma,\ \mathbf{r}_0, \tag{1}$$
$$\tilde{F}^{(\chi)}(\mathbf{r}, t) = \frac{b^{(\chi)}}{r}\delta(\mathbf{r} - \mathbf{r}_0)f(t - t_0);$$

the initial condition

$$\upsilon^{(\chi)} \equiv 0 \equiv \frac{\partial \upsilon^{(\chi)}}{\partial t}, \quad t \le t_0; \tag{2}$$

the boundary condition

$$\frac{\partial^{\chi-1}}{\partial n^{\chi-1}}\left(\frac{\partial \upsilon^{(\chi)}}{\partial t}\right)\bigg|_{\Sigma} = 0; \tag{3}$$

the bounded-energy condition

$$\iiint_V \left(\left|\frac{\partial \upsilon^{(\chi)}}{\partial t}\right|^2 + \left|\nabla \upsilon^{(\chi)}\right|^2\right) dV < \infty. \tag{4}$$

Here, the superscript $\chi = 1, 2$ determines the type of the source surface. According to [6, 7], the boundary value problem given by Eqs. (1)–(4) has a unique solution. We represent the potential $\upsilon^{(\chi)}(\mathbf{r}, t)$ as

$$\upsilon^{(\chi)}(\mathbf{r}, t) = \upsilon_0^{(\chi)}(\mathbf{r}, t) + \upsilon_1^{(\chi)}(\mathbf{r}, t),$$

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where

$$\upsilon_0^{(\chi)} = -\frac{b^{(\chi)}}{4\pi r_0 R} f\left(t - t_0 - \frac{1}{a}R\right)\eta\left(t - t_0 - \frac{1}{a}R\right).$$

The source potential $\upsilon_1^{(\chi)}(\mathbf{r}, t)$ is the desired Debye potential that corresponds to the field excited by the source, $\eta(\xi)$ is the Heaviside step function, and $R = |\mathbf{r} - \mathbf{r}_0|$.

GREEN's FUNCTION AND THE MELER–FOCK INTEGRAL TRANSFORMATIONS

We express the potential $\upsilon^{(\chi)}(\mathbf{r}, t)$ in terms of the Green's function and use it to solve the boundary value problem given by Eqs. (1)–(4):

$$\upsilon^{(\chi)}(\mathbf{r}, t) = \frac{b^{(\chi)}}{r_0}\int_0^{t-t_0} G^{(\chi)}(\mathbf{r} - \mathbf{r}_0, z) f(t - t_0 - z)dz. \quad (5)$$

The Green's function is

$$G^{(\chi)}(\mathbf{r}, t) = G_0(\mathbf{r}, t) + G_1^{(\chi)}(\mathbf{r}, t), \quad (6)$$

where

$$G_0(\mathbf{r}, t) = \frac{\delta\left[t - t_0 - \frac{R}{a}\right]}{4\pi R}$$

is the Green's function of the free space, satisfying wave equation (1) having the δ-shaped right-hand side, initial condition (2), boundary condition (3), and boundedness condition (4). We seek the potential $\upsilon_1^{(\chi)}(\mathbf{r}, t)$ in the form of (5), whereas the initial problem is reduced to finding the function $G_1^{(\chi)}(\mathbf{r}, t)$ for the complicated conic structure Σ. To this end, we use the Laplace transformation for the function $G^{(\chi)}(\mathbf{r}, t)$ with respect to the time parameter

$$G^{s,(1)} = G^{s,(1)}(\mathbf{r}) = \int_0^{+\infty} G^{(1)}(\mathbf{r}, t)e^{-st}dt, \quad \mathrm{Re}\, s > 0. \quad (7)$$

We find the image $G^{s,(1)}$ that must satisfy the following requirements:

the inhomogeneous Helmholtz equation

$$(\Delta - q^2)G^{s,(\chi)}(\mathbf{r}) = -e^{-st_0}\delta(\mathbf{r} - \mathbf{r}_0),$$
$$\mathbf{r} \notin \Sigma_0, \quad \mathbf{r}_0, \quad q = \frac{s}{a}; \quad (8)$$

the boundary condition

$$\frac{\partial^{\chi-1}}{\partial n^{\chi-1}} G^{s,(\chi)}|_\Sigma = 0; \quad (9)$$

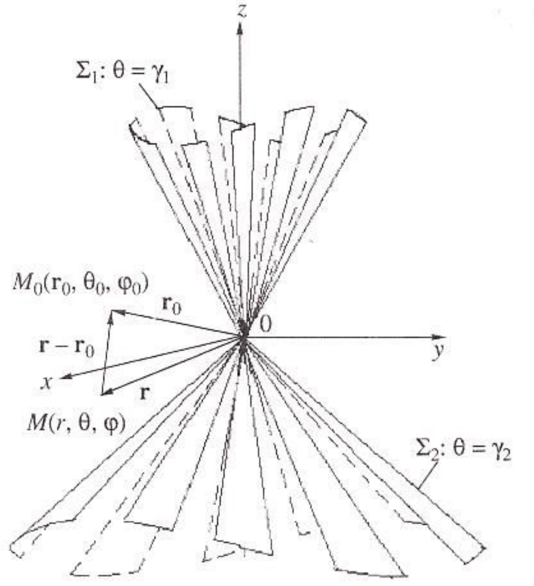

Fig. 1. Structure geometry.

the principle of the ultimate absorption;

and the bounded-energy condition.

We assume that $q > 0$. Then, we perform the analytical extension in finite formulas. In accordance with (6), we write $G^{s,(1)}$ in the form

$$G^{s,(\chi)}(\mathbf{r}) = G_0^s(\mathbf{r}) + G_1^{s,(\chi)}(\mathbf{r}),$$

where

$$G_0^s(\mathbf{r}) = \frac{e^{-(qR + st_0)}}{4\pi R} = \frac{2}{\pi^2}\int_0^{+\infty} \tau\sinh\pi\tau\,\hat{G}_0^s \frac{K_{i\tau}(qr)}{\sqrt{r}}d\tau,$$

$$\hat{G}_0^s = \sum_{m=-\infty}^{+\infty} \hat{\alpha}_{m\tau}^s \hat{U}_{m\tau}^{(0)} e^{im\varphi},$$

$$U_{m\tau}^{(0)}(\theta, \theta_0) = \begin{cases} P_{-1/2+i\tau}^m(\cos\theta)P_{-1/2+i\tau}^m(-\cos\theta_0), \\ \theta < \theta_0 \\ P_{-1/2+i\tau}^m(-\cos\theta)P_{-1/2+i\tau}^m(\cos\theta_0), \\ \theta_0 < \theta, \end{cases} \quad (10)$$

$$\hat{\alpha}_{m\tau}^s = \frac{(-1)^m}{4\cosh\pi\tau}\frac{K_{i\tau}(qr_0)}{\sqrt{r_0}}\frac{\Gamma\left(\frac{1}{2} - m + i\tau\right)}{\Gamma\left(\frac{1}{2} + m + i\tau\right)}e^{-i(st_0 + m\varphi_0)}.$$

Here, $K_{i\tau}(qr)$ is the modified Bessel function of the second kind; $P^m_{-1/2+i\tau}(\cos\theta)$ is the associated Legendre function of the first kind; and $\Gamma(z)$ is the gamma function. To solve problem (8), (9), we exploit the integral Kontorovich–Lebedev representation:

$$\hat{g}(\tau) = \int_0^{+\infty} g(r)\frac{K_{i\tau}(qr)}{\sqrt{r}}dr, \quad (11)$$

$$g(r) = \frac{2}{\pi^2}\int_0^{+\infty} \tau\sinh\pi\tau\hat{g}(\tau)\frac{K_{i\tau}(qr)}{\sqrt{r}}d\tau. \quad (12)$$

We now represent $G_1^{s,(1)}(\mathbf{r})$ in the form of (12). In this case, we have

$$G_1^{s,(\chi)}(\mathbf{r}) = \frac{2}{\pi^2}\int_0^{+\infty} \tau\sinh\pi\tau\hat{G}_1^{s,(\chi)}\frac{K_{i\tau}(qr)}{\sqrt{r}}d\tau; \quad (13)$$

$$\begin{gathered}\hat{G}_1^{s,(1)} = \sum_{m=-\infty}^{+\infty}\hat{b}^s_{m\tau}\hat{U}^{(1)}_{m\tau}(\theta,\varphi),\\ \hat{b}^s_{m\tau} = -\hat{\alpha}^s_{m\tau}P^m_{-1/2+i\tau}(-\cos\theta_0)P^m_{-1/2+i\tau}(\cos\gamma_2),\\ \gamma_2<\theta_0;\end{gathered} \quad (14)$$

$$\hat{U}^{(\chi)}_{m\tau} = \begin{cases}\displaystyle\sum_{n=-\infty}^{+\infty}\hat{\alpha}^{(\chi)}_{mn}P^{m+nN}_{-1/2+i\tau}(\cos\theta)e^{i(m+nN)\varphi},\\ 0<\theta<\gamma_1\\ \displaystyle\sum_{n=-\infty}^{+\infty}[\hat{\beta}^{(\chi)}_{mn}P^{m+nN}_{-1/2+i\tau}(\cos\theta)\\ +\hat{\xi}^{(\chi)}_{mn}P^{m+nN}_{-1/2+i\tau}(-\cos\theta)]e^{i(m+nN)\varphi}, \quad \gamma_1<\theta<\gamma_2\\ \displaystyle\sum_{n=-\infty}^{+\infty}\hat{\zeta}^{(\chi)}_{mn}P^{m+nN}_{-1/2+i\tau}(-\cos\theta)e^{i(m+nN)\varphi},\\ \gamma_2<\theta<\pi.\end{cases} \quad (15)$$

Here, $\hat{\alpha}^{(\chi)}_{mn}$, $\hat{\beta}^{(\chi)}_{mn}$, $\hat{\xi}^{(\chi)}_{mn}$, and $\hat{\zeta}^{(\chi)}_{mn}$ are unknown coefficients independent of the parameter q. Based on the results of [3, 8], one can show that the Green's function $G^{(\chi)}_\kappa(\mathbf{r}, t)$ (6) ($\kappa = 0, 1$) can be represented (in a unique manner) as the integral

$$G^{(\chi)}_\kappa(\mathbf{r}, t) = \int_0^{+\infty}\tau\tanh\pi\tau\tilde{\tilde{G}}^{(\chi)}_\kappa P_{-1/2+i\tau}(\cosh b)d\tau, \quad (16)$$

where

$$\tilde{\tilde{G}}_0 = \frac{1}{r}\eta\left(t-t_0-\frac{r+r_0}{a}\right)\sum_{m=-\infty}^{+\infty}a_{m\tau}U^{(0)}_{m\tau}e^{im\varphi}, \quad (17)$$

$$\begin{gathered}\tilde{\tilde{G}}^{(\chi)}_1 = -\frac{1}{r}\eta\left(t-t_0-\frac{r+r_0}{a}\right)\sum_{m=-\infty}^{+\infty}a_{m\tau}\hat{U}^{(\chi)}_{m\tau}\\ \times P^m_{-1/2+i\tau}(-\cos\theta_0)P^m_{-1/2+i\tau}(\cos\gamma_2),\end{gathered}$$

$$a_{m\tau} = \frac{1}{4\pi r_0}(-1)^m e^{-im\varphi_0}\frac{\Gamma\left(\frac{1}{2}-m+i\tau\right)}{\Gamma\left(\frac{1}{2}+m+i\tau\right)}, \quad (18)$$

$$\cosh b = \frac{a^2(t-t_0)^2-r^2-r_0^2}{2rr_0}, \quad b\in[0,+\infty).$$

The integral representation of type (16) is a version of the Meler–Fock integral representation [4], which can be written in the form

$$\Psi(b) = \int_0^{+\infty}\tau\tanh\pi\tau\hat{\Psi}(\tau)P_{-1/2+i\tau}(\cosh b)d\tau, \quad (19)$$

where

$$\hat{\Psi}^{(\tau)} = \int_0^{+\infty}\sinh b\Psi(b)P_{-1/2+i\tau}(\cosh b)db, \quad b\in[0,+\infty). \quad (20)$$

The Green's function $G^{(\chi)}_1(\mathbf{r}, t)$ (6) for a complicated conic structure Σ can be found by using integral transformations (19), (20), representations (17), (18), boundary condition (3), and the conjugation condition in slots. As a result, we arrive at two coupled sets of functional equations for the determination of unknown coefficients of the function $\hat{U}^{(\chi)}_{m\tau}$ (15). The form of these sets is presented in [3, 9], and their solution can be obtained employing the method of singular integral equations or the method of the Riemann–Hilbert problem [3, 5]. We present expressions describing the Debye potentials in certain particular cases of a complicated conic structure and the function $f(t-t_0)$.

(A). $f(t) = e^{i\alpha\omega t}$, $\alpha = \pm1$, $t_0 = 0$.

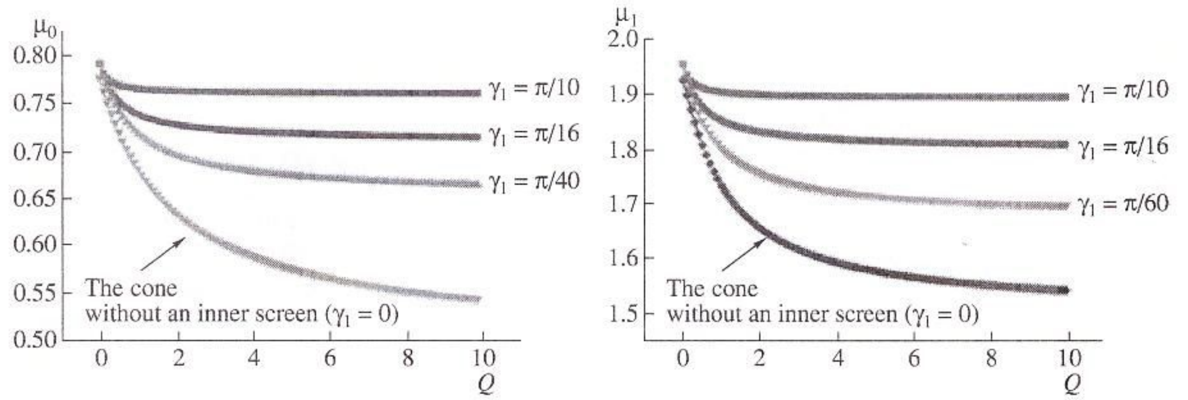

Fig. 2. Dependence of the function $\mu_n(Q, \gamma_1, \gamma_2)$ on the filling parameter Q for different angles γ_1; $\gamma_2 = \frac{\pi}{8}$.

The single cone Σ_1 with N narrow slots $\left(\chi = 2, \frac{d_1}{l} \ll 1, \theta_0 = \pi\right)$:

$$\upsilon_1^{(2)} = \upsilon_{1\,\text{solid}}^{(2)} + \frac{1}{-\frac{1}{N}\ln\left(\frac{1-u_1}{2}\right)}\int_0^\infty a_{i\tau}^{*(2)}\frac{K_{i\tau}(\tilde{q}r)}{\sqrt{r}}\frac{F_{i\tau}^*}{\tilde{W}_{i\tau}^{(2)}}$$

$$\times A_{i\tau}^{*(2)}P_{-1/2+i\tau}(-\cos\theta)d\tau + \frac{1}{-\frac{1}{N}\ln\left(\frac{1-u_1}{2}\right)}$$

$$\times\int_0^\infty a_{i\tau}^{*(2)}\frac{K_{i\tau}(\tilde{q}r)}{\sqrt{r}}\frac{F_{i\tau}^*}{\tilde{W}_{i\tau}^{(2)}}A_{i\tau}^{*(2)}$$

$$\times\sum_{n\neq 0}\frac{1}{1-\frac{|n|}{n}\varepsilon_{n,2}^{*(2)}V_{n-1}^{n-1}(u_1)}\frac{P_{-1/2+i\tau}^{nN}(-\cos\theta)}{\frac{d}{d\gamma_2}P_{-1/2+i\tau}^{nN}(-\cos\gamma_2)}$$

$$\times e^{inN\varphi}d\tau + O(1-u_1), \quad \gamma_1 < \theta < \pi, \tag{21}$$

$$\tilde{W}_{i\tau}^{(2)} = F_{i\tau}^* + \frac{1}{-\frac{1}{N}\ln\left(\frac{1-u_1}{2}\right)},$$

$$F_{i\tau}^* = \frac{1}{A_{i\tau}^{*(2)} - \frac{1}{N}\sum_{p\neq 0}\frac{1}{|p|}\varepsilon_{p,2}^{*(2)}}.$$

Here, $u_1 = \cos\left(\frac{\pi d_1}{l}\right)$, $\tilde{q} = i\alpha k$, k is the wave number; $a_{i\tau}^{*(2)}$, $\varepsilon_{n,\tau}^{*(2)}$, and $A_{i\tau}^{*(2)}$ are the known coefficients; the functions $V_{n-1}^{n-1}(u_1)$ are defined in [10]; and $\upsilon_{1\,\text{solid}}^{(2)}$ are the potentials for the solid cone Σ_1 [11]. Representation (21) is valid far from the slots.

(B). $f(t-t_0) = \delta(t-t_0)$, $\chi = 1$, $\theta_0 = \pi$.

The conic structure Σ consists of a solid cone Σ_1 and a semitransparent cone Σ_2. The latter is determined by the existence of the limit

$$Q = \lim_{\substack{N\to+\infty \\ d_2/l\to 1}}\left[-\frac{1}{N}\ln\cos\frac{\pi d_2}{2l}\right],$$

$$\upsilon_1^{(1)} = \eta\left(t - t_0 - \frac{r+r_0}{a}\right)\frac{Q}{r}\sum_{n=0}^{+\infty}\psi_{\mu_n}(\gamma_1,\gamma_2)$$

$$\times P_{-1/2+\mu_n}(-\cos\theta)Q_{-1/2+\mu_n}(\cosh b), \quad \gamma_2 < \theta < \pi, \tag{22}$$

$$\psi_\mu(\gamma_1,\gamma_2) = -\frac{ab^{(1)}}{\pi r_0^2}\mu\cos\pi\mu$$

$$\times\frac{\Delta_\mu(\gamma_1,\gamma_2)}{P_{-1/2+\mu}(-\cos\gamma_1)\frac{d}{d\mu}\nabla_\mu}\frac{P_{-1/2+\mu}(\cos\gamma_2)}{P_{-1/2+\mu}(-\cos\gamma_2)},$$

$$\nabla_\mu = \pi\frac{P_{-1/2+\mu}(-\cos\gamma_2)}{P_{-1/2+\mu}(-\cos\gamma_1)}\Delta_\mu^{(1),0}(\pi-\gamma_1,\pi-\gamma_2) + 2Q\cos\pi\mu, \quad \nabla_{\mu_n} = 0, \tag{23}$$

where $Q_{-1/2+\mu}(\cosh b)$ is the Legendre function of the second kind. Representation (22) is obtained from (16) as a result of the expansion of the integral (after the passage in (16) to integration over the imaginary axis $\mu = i\tau$) into a series in terms of residues of the integrand in its simple poles $\mu_n(Q, \gamma_1, \gamma_2)$, in which are contained the roots of Eq. (23). Figure 2 exhibits the values μ_0, μ_1 as functions of the filling parameter Q for $\gamma_2 = \frac{\pi}{8}$ and different angles γ_1. The sequence of values $\{\mu_n\}_{n=0}^{+\infty}$ is monotonically increasing. The least of the values μ_0 determines the field behavior near the structure vertex and the field spatial distribution in the case when the source is closely located to the vertex.

Thus, we have proposed and substantiated a method for solving boundary value problems of diffraction in the time region for complicated three-dimensional unclosed conic structures. The method constitutes a generalization of the results reported by the authors in [3, 9] as applied to solving problems of wave diffraction on unclosed perfectly conducting bicones in the frequency region. This approach can also be used in solving time-dependent problems of wave diffraction on three-dimensional impedance structures of conic geometry.

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