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Obtaining the Local Extremum in the Problem of Covering the Fields by the Circles of Variable Radius

Рассмотрена задача покрытия области кругами переменных радиусов. Построена математическая модель покрытия. Предложен новый критерий покрытия, на основании которого аналитически описана область допустимых решений задачи. Исходя из анализа свойств модели, показано, что решение задачи может быть сведено к решению последовательности задач нелинейного программирования.

Ключевые слова: покрытие кругами переменных радиусов, критерий покрытия, оптимизация, нелинейное программирование.

The problem of covering the area of the variable radius circle is considered. The mathematical model of the coating is built. A new coverage criterion is offered, based on which the range of permissible solutions of the problem is analytically described. Based on the analysis of the model properties, it is shown that the solution of the problem can be reduced to the nonlinear programming sequence solution of problems.

Keywords: circle of variable radius coverage, coverage criteria, optimization, nonlinear programming.

Розглянуто задачу покриття області колами змінних радіусів. Побудовано математичну модель покриття. Запропоновано новий критерій покриття, на підставі якого аналітично описано область допустимих розв'язань задачі. Виходячи з аналізу властивостей моделі, показано, що розв'язання задачі може бути зведено до розв'язання послідовності задач нелінійного програмування.

Ключові слова: покриття колами змінних радіусів, критерій покриття, оптимізація, нелінійне програмування.

Formulation of the problem. In various industries of the economy there are the problems associated with the processing and transformation of geometric information. These questions are referred to a class of the optimization geometric design problems [1], which solution as well as the development of their methods is important. This class of problems includes the matter of optimal material cutting (both regular and irregular), the problems of building the optimal ways and linking networks, coverage, partition, some scheduling, and others. [1–2].

Analysis of publications

An important class of geometric design problems are the matters of an irregular covering the field by geometric objects [3], as well as regular [3–7]. In the problems of covering, it is set up a claim that all points of the field were covered by geometric objects, while the conditions of non-intersection of objects between themselves and their placement in the field may be violated. The results and detailed reviews on given researches are in [8–10]. The problems of single covering a limited area by *N*-circles (such as in a Euclidean metric, and in some other metrics) is also known as the problem of *N*-centers. For this in different metrics it is offered a variety of heuristics and algorithms using Voronoy's regions [11]. The problems of coverage are the models of many practical problems. In [12] it is set and solved the problem of interaction between militarized security subdivisions of the railroad and fire-rescue units, which is reduced to covering the area by the circles of different radii. In [13] it is set the problem of the placement optimization for the observation points, which arises, when designing ground video monitoring systems. The problem is reduced to the question of covering the area by the circles of variable radii, the value of which depends on the class of fire danger and its relief. Thus, the important practical problems require the developing the methods of covering the fields by circles of variable radius.

In our article, we propose an approach to obtaining a local extremum of covering problem.

Setting the problem

There is a polygon P, defined by a set of vertices p_k, $k=1,2,...,n$ and a set of circles C_i, $i=1,2,...,N$, with varying radius $r_i<r$ and centers $v_i=(x_i, y_i)$.

Suppose $u=(v_1,r_1,...,v_N,r_N)$ – vector of variables, $F(u)$ – the objective function (for example, the total area of the circles $\pi\sum_{i=1}^{N} r_i^2$),

$\Xi(u)=\bigcup_{i=1}^{N} C_i(u_i, r_i)$.

By a definition $\Xi(u)$ – coverage of polygon P if $P \subset \Xi(u) \Leftrightarrow \Xi^{'}(u)\bigcap P=\varnothing$, where $\Xi^{'}(u)= = R^2 \setminus \operatorname{int}\Xi(u)$.

Note 1. In this study, we consider only such coverings, which met the following conditions $\operatorname{int} C_i \not\subset \Xi_i(u)$, $\Xi_i(u)=\bigcup_{j=1}^{i-1} C_j(v_j, r_j)\bigcup_{k=i+1}^{N} C_k(v_k, r_k)$ and $\operatorname{int} C_i(v_i, r_i)\bigcap \operatorname{int} P \neq \varnothing$, $i=1,2,...,N$.

Problem of circular coverage of polygon. The start point – vector $u^0=(v_1^0, r_1^0, ..., v_N^0, r_N^0)$, where $\Xi(u^0)$ covers a polygon *P*. The task – to determine the vector $u^*=(v_1^*, r_1^*, ..., v_N^*, r_N^*)$, in which $F(u)$ reaches the extreme and $\Xi(u^*)$ is coverage of a polygon *P*.

The problem of covering polygon by circles

Mathematical model of the circular coverage problem can be represented as follows:

$$\operatorname*{extr}_{u\in W} F(u), \qquad (1)$$

where

$$W=\{u\in R^{3N} : \Xi^{'}(u)\bigcap P=\varnothing\}. \qquad (2)$$

As a criterion of covering for a fixed u can be used phi-function method[14]:

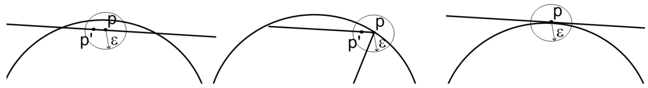
$$\Xi^{'}(u)\bigcap P=\varnothing \Leftrightarrow \Phi^{\Xi^{'}P} \geq 0,$$

where $\Phi^{\Xi^{'}P}$ – phi-function of objects $\Xi^{'}(u)$ and *P* [14].

Since the description of admitted region of the form (2) in an analytical form is extremely difficult theoretical problem and requires significant computational cost, in this study we propose the coverage criteria based on the following statement.

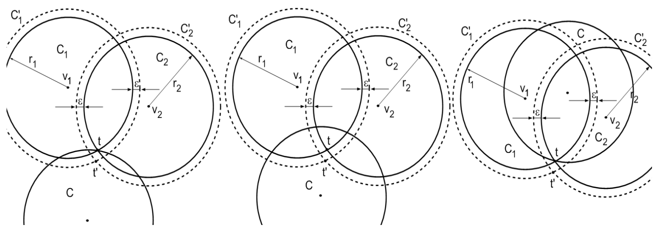
We say that a relative position of the circle *C* and the points $p \in frP$ satisfies the condition 1, if $p \in C$ and for any arbitrarily small $\varepsilon > 0$ in ε-neighborhood of the point *p* there is a point $p^{'}$ such that $p^{'} \in frP$ and $p^{'} \in C$. Here frP – the boundary of the set *P* (Figure 1.).

We say that the relative position of the intersection point *t* of circle C_1 and C_2 with the centers in the points v_1 and v_2 radius r_1 and r_2 with the circle *C* satisfies the condition 2, if $t \in C$ and there is a positive number δ such that for any arbitrarily small strictly positive $\varepsilon < \delta$ intersection point $t^{'}$ of circles $C_1^{'}$ and $C_2^{'}$ and radius $r_1+\varepsilon$ and $r_2+\varepsilon$ with centers in points v_1 and v_2 also belongs to the circle C (Figure 2).



a *b* *c*

Fig. 1. *a* and *b* – the relative position of the point and the circle satisfies the condition 1; *c* – the relative position of the point and the circle does not satisfy the condition 1



a *b* *c*

Fig. 2. *a* and *b* – the relative position of the point and the circle satisfies condition 2; *c* – the relative position of the point and the circle does not satisfy the condition 2

Note that if $t \in \operatorname{int} C$, the relative position of the point of intersection *t* and the circle *C* satisfies the second condition.

Statement. In order $\Xi(u)$ to cover the polygon *P*, it is necessary and sufficient that the vector $u=(v_1, r_1, ..., v_N, r_N)$ satisfies the condition:

1) for each vertex p_k of the polygon *P* there is at least one circle C_i such that their relative position satisfies the first condition (Figure 3);

2) for any point $t \in frC_i \bigcap frP$, the relative position with the circle C_i satisfies the first condition, there is a circle $C_j, i \neq j$, which the relative position of the point also satisfies the first condition (Figure 4);

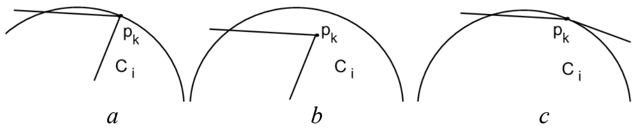


Fig. 3. Three types of the relative position of the vertex of the field and the circle, which satisfy the first item

Fig. 4. Two types of relative position of the intersection point of the circle with the field and the circle for the item 2)

3) for each point $t = frC_i \bigcap frC_j$, $t \in frP$ exists $C_s, s \neq i, s \neq j$, such that the relative position of the point *t* and the circle C_s satisfies the second condition (Figure 5);

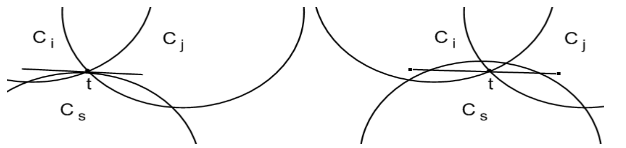


Fig. 5. Two types of relative position of the intersection point of circles and the circle for the item 3)

4) for each point $t_q = frC_i \bigcap frC_j$, t_q , $q = 1,2$ $i \neq j$, there are $C_s, s \neq i, s \neq j$, such that the relative position t_q and C_s satisfies the second condition (Figure 6).

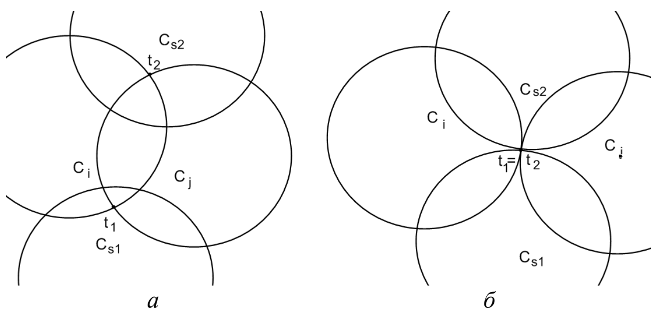


Fig. 6. Two types of relative position of the intersection point of circles and the circles for the item 4)

Note 2. Further when forming set of constraints for the problem to item 4, we do not take into account two circles for which both points t_q , $q = 1,2$, belong to inner part of the circle C_s , since taking into account these conditions appear to be excessive.

Note 3. It is assumed that for item 4 the points t_q , $q = 1,2$ are different and even in the case of their superposition (circles C_i and C_j are contiguous) each of them belongs to its circle (Figure 6*b*).

Items 1 – 4 can be rewritten in the equivalent form

1) $\forall p_k \ k \in I_n$, exists such circle C_i, that $p_k \in C_i$, $C_i \not\subset P$.

2) If there is a point $t \in frC_i \bigcap frP$, $C_i \not\subset P$, then there are a circle C_j and a point $v_{ij}, i \neq j$, such that $v_{ij} \in C_i$, $v_{ij} \in C_j$, $v_{ij} \in R^2 \setminus \operatorname{int} P$;

3) if there is a point $t = frC_i \bigcap frC_j$ and wherein $t \in frP$, there is C_s and a point v_{its}, $s \neq i, s \neq j$, such that $v_{ijs} \in C_i$, $v_{ijs} \in C_j$, $v_{ijs} \in C_s$;

4) If there is a point $t_q = frC_i \bigcap frC_j$ and $t_q \in \operatorname{int} P$, $q = 1,2$, $i \neq j$, then there is a circle C_s and a point $v_{ijsq}, s \neq i, s \neq j$, such, that $v_{ijsq} \in C_i$, $v_{ijsq} \in C_j$, $v_{ijsq} \in C_{sq}$.

On the figure 7 there is an example of a polygon coverage by the set of circles with the points of type $v_{ij}, v_{ijs}, v_{ijs1}, v_{ijs2}$.

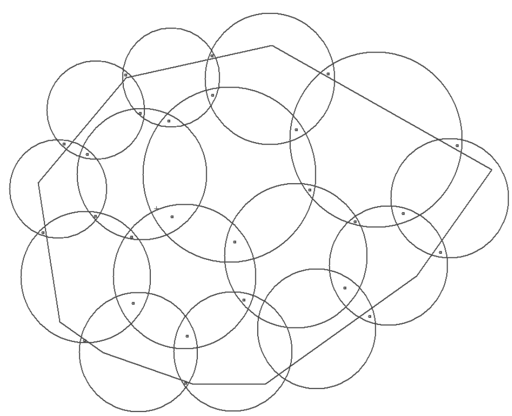


Fig. 7. Example of coverage by circles with system of auxiliary points

With this in mind, the inequalities describing the admitted region of the problem based on the information about the start point can be written as:

1) for the vertices of the polygon $\forall p_k\ k \in I_n$, and the corresponding circles C_i, $p_k \in C_i$, $C_i \not\subset P$, inequality:

$$(x_i - x_k)^2 + (y_i - y_k)^2 \le r_i^2, \qquad (3)$$

2) for points $t \in frC_i \bigcap frP$, circles $C_i \not\subset P$ and the corresponding circles C_j – the inequalities system in the form

$$\begin{cases} (x_i - x_{ij})^2 + (y_i - y_{ij})^2 \le r_i^2; \\ (x_j - x_{ij})^2 + (y_j - y_{ij})^2 \le r_j^2; \\ f_{ij}(x_{ij}, y_{ij}) \ge 0, \end{cases} \qquad (4)$$

where $f_{ij}(x_{ij}, y_{ij}) \ge 0$ – the membership function of the set $R^2 \setminus \operatorname{int} P$ of points v_{ij} (maximum of k linear functions);

3) for points $t = frC_i \bigcap frC_j$, $t \in frP$ and the corresponding circles C_s – the inequalities system in the form

$$\begin{cases} (x_i - x_{ijs})^2 + (y_i - y_{ijs})^2 \le r_i^2; \\ (x_j - x_{ijs})^2 + (y_j - y_{ijs})^2 \le r_j^2; \\ (x_s - x_{ijs})^2 + (y_s - y_{ijs})^2 \le r_s^2; \end{cases} \qquad (5)$$

4) for points $t_q = frC_i \bigcap frC_j$ and $t_q \in \operatorname{int} P$, $q = 1,2$, $i \ne j$ and the corresponding circles C_{sq} – the inequalities system in the form:

$$\begin{cases} (x_i - x_{ijs1})^2 + (y_i - y_{ijs1})^2 \le r_i^2; \\ (x_j - x_{ijs1})^2 + (y_j - y_{ijs1})^2 \le r_j^2; \\ (x_{s1} - x_{ijs1})^2 + (y_{s1} - y_{ijs1})^2 \le r_{s1}^2; \\ f(x_{ijs1}, y_{ijs1}) < 0, \end{cases} \qquad (6)$$

where $f(x_{ijsq}, y_{ijsq}) < 0$ – membership function of the point $v_{ijsq}, q = 1,2$ set P and

$$\begin{cases} (x_i - x_{ijs2})^2 + (y_i - y_{ijs2})^2 \le r_i^2; \\ (x_j - x_{ijs2})^2 + (y_j - y_{ijs2})^2 \le r_j^2; \\ (x_{s1} - x_{ijs2})^2 + (y_{s1} - y_{ijs2})^2 \le r_{s1}^2; \\ f(x_{ijs2}, y_{ijs2}) < 0. \end{cases} \qquad (7)$$

It is necessary to find an extremum of the given objective function at admitted region W, given by constraints systems in the form (3) – (7).

The problem belongs to non-smooth optimization problems due to the presence of non-smooth functions belonging to the type $f_{ij}(x_{ij}, y_{ij}) \ge 0$ (4) and $f(x_{ijsq}, y_{ijsq}) < 0$ in (6) and (7). Admitted region can be divided into sub-areas, described by the inequality systems with the smooth functions. Thus, the solution of the problem can be reduced to solving the sequence of tasks of nonlinear programming. As the objective function can be chosen an arbitrary smooth function, including: minimizing the radii of the covering circles, minimizing the number of circles (if it is possible to reduce one of the radii down to zero) and the reliability (maximization areas of overlap).

Figure 8 – 13 shows examples of test problems of covering a circle by the circles, where local optimization was produced from the obtained points with the different objective functions.

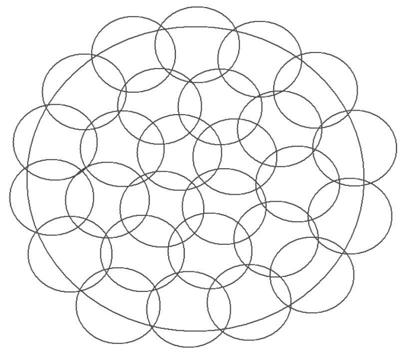


Fig. 8. Original cover (produced by hand) of the field with radius 4, covering 27 circles, radii of the circles is 1

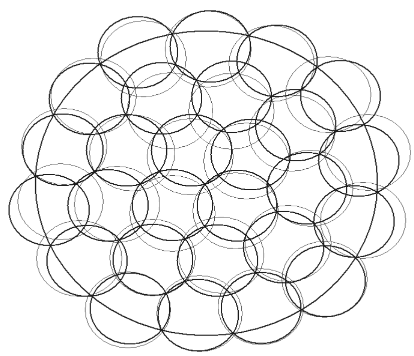


Fig. 9. Local extremum with objective function – the minimization of the maximum radius of the covering circles. Time of calculating 0,09 s., the radius is 0,941

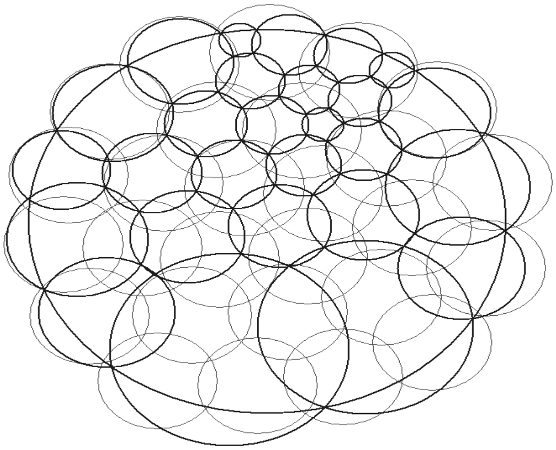


Fig. 10. Local extremum with objective function – the minimization of the radii sum of the covering circles. Time of calculating is 0,125 s., the sum is equal to 22,942

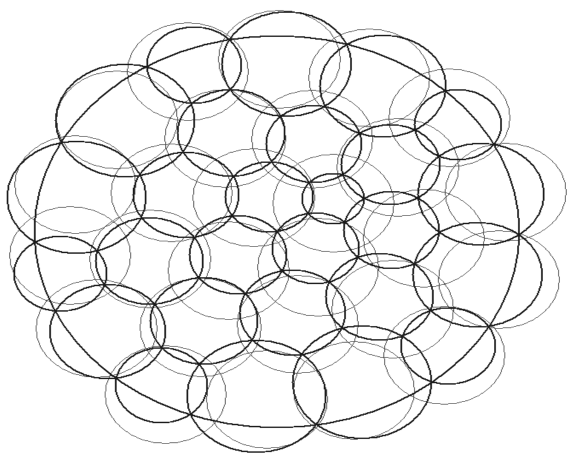


Fig. 11. Local extremum with objective function – the minimization of the squares of the radii sum for the covering circles. Time of calculating is 0,047 s., the sum of the radii squares is 22,106

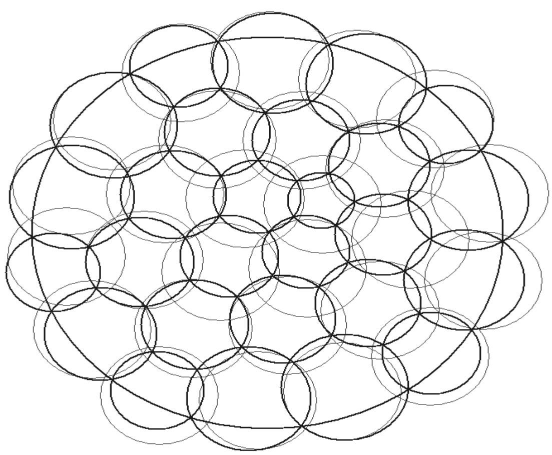


Fig. 12. Local extremum with objective function – the minimization of the cubes of the radii sum for the covering circles. Time of calculating is 0,062 s., the sum of cubic radii is equal to 20,693

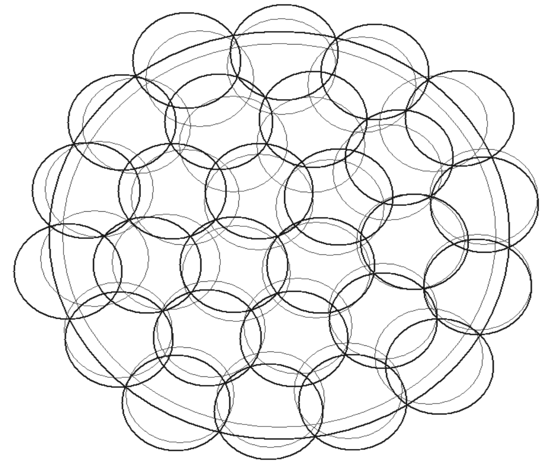


Fig. 13. Local extremum with objective function – maximizing the radius of the field at the fixed radii of the circles. Time of calculating is 0,093 s., radius of the field is 4,251

Conclusions. The constructed model of covering the polygon by the circles of variable radii and a method for obtaining the local extremum can be used for a wide range of the practical questions, in particular for the problem of locating points of video surveillance, when designing ground video monitoring systems.

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Окончание статьи Н.К. Тимофеевой

Так, сходимость последовательности решений, построенных методом структурно-алфавитного поиска для задачи коммивояжера приближается к нулю, т.е. $\Delta(F_{\min}, F(w^{k^*})) \to 0$, а для подклассов разрешимых задач $\Delta(F_{\min}, F(w^{k^*})) = 0$. Ее скорость – полиномиальна по вычислительной сложности. Сходимость метода ближайшего соседа и «жадного» алгоритма зависит от структуры входных данных. Для одних структур решение равно $\Delta(F_{\min}, F(w^{k^*})) = 0$, а для других может быть $\Delta(F_{\min}, F(w^{k^*})) \to 1$. Это решение, как правило, достигается за одну итерацию.

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N.K. Tymofijeva

The Proof of the Algorithms Convergence for Combinatorial Optimization with the Using Subclasses of the Solved Problems

Keywords: combinatorial optimization, combinatorial configuration, objective function, traveling salesman problem, structure-alphabetical search method, nearest neighbor method, «greedy» algorithm.

The proof of the approximate solutions convergence sequence to a global solution of the combinatorial optimization problem, which is based on a particular algorithm, is rather complicated problem. This is due to the fact that some classes of problems are unsolvable because of their computational complexity. A lot of researches are devoted to the problem of the methods and algorithms convergence within the mathematical programming. They enter the formal level features required and sufficient conditions for their convergence.

The original way to prove some convergence combinatorial optimization methods, based on recognition of the structure of input data (structure-alphabetical search method nearest neighbor method, "greedy" algorithm) is presented. For this purpose, the subclasses of the traveling salesman problems is used. A sequence of the convergence solutions that are built specifically is proved.

To assess the methods accuracy, which are decided on a set of permutations, the input data of combinatorial optimization problems defines the functions of the natural argument, one of which is combinatorial. This allows to define a set of values of the objective function for basic problem and to establish some error of interpretation algorithm.

An solvable case for the traveling salesman problem is shown, in which the input data requires the linear combinatorial function for which analytically the global minimum and maximum are found. Using this case proves that the convergence of a solutions sequence built by the structural alphabet search for the traveling salesman problem is close to zero. The optimal solution for subclasses coincides with the global. The speed of the described method is polynomial of computational complexity. The convergence of the nearest neighbor method and of the "greedy" algorithm depends on the structure of input data. For some, the solution structures coincides with the global, while others may be far from optimal.

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