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ON THE CONSTRUCTION OF TWO-SIDED APPROXIMATIONS TO POSITIVE SOLUTIONS OF SOME ELLIPTIC PROBLEM

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Abstract: In this paper we have investigated the existence, uniqueness and possibility of constructing of two-sided approximations to the positive solution of a heat conduction problem with two sources.

The investigation is based on methods in operator equations theory in half-ordered spaces. In this case we have considered a nonlinear operator equation that corresponds to the initial boundary value problem in a cone of non-negative continous functions. The properties of the corresponding operator define conditions which provide the existence and uniqueness of the solution. The conditions link the parameters of the problem implicitly meaning that they don't provide the range of allowed values but need to be verified for each specific parameters value set separately.

During the investigation we have provided the scheme of a two-sided iteration process which must satisfy the conditions in order to converge to the positive solution from both sides.

The computational experiment have been conducted in two domains – unit disk and unit half disk. We have applied both two-sided approximations method and Green's quasifunction method for the problem solving. The obtained results are presented as a surface and level lines plots and also as a table. The results in corresponding domains obtained by different methods have been compared with each other.

Key words: two-sided approximations, operator equation, positive solution, concave operator, conical interval, Green's function, Green's quasifunction.

INTRODUCTION

Modern science is highly interested in processes that take place in nonlinear environments. Mathematical models of these processes typically are represented by nonlinear boundary value problems of mathematical physics of the following form

$$-\Delta u = f(\lambda, u) \quad \forall x \in \Omega \subset R^n, \qquad (1)$$

$$u > 0, \quad u|_{\partial\Omega} = 0, \qquad (2)$$

where: λ is a numerical parameter.

Many profound problems are reduced to equation (1). For example:

1) various problems in the theory of elasticity, where the parameter represents the load;

2) temperature distribution during conduction of electrical current through a body (the parameter is a value of electrical current);

3) auto-oscillation problems (the parameter is the unknown period) etc.

More specifically, if

$$f(u) = e^{-u},$$

then problem (1), (2) is a mathematical model of a flow in conductive environment inside an impenetrable cylinder [1]; when

$$f(u) = \lambda e^{u},$$

equation (1) is a stationery equation of the thermal theory of spontaneous ignition of chemically active gas mixture inside a vessel [2-5], in this case problem (1), (2) is called the Liouville-Gelfand problem; if

$$f(u) = u^p, \ p > 0,$$

then we have a mathematical model of gas density distribution in a star (equation (1) in that case is called the Lane–Emden equation) [6]; the problem of model selection of population migration in genetics leads to problem (1), (2) with

$$f(u) = \lambda(1+u)^q \quad [7];$$

problem (1), (2) with

$$f(u) = \lambda + u^p, \ f(u) = \lambda u^q + u^p,$$
$$f(u) = \lambda\left(e^u + \ + e^{\gamma u}\right)$$

are considered in [8, 9] and with

$$f(u) = au^{-q} + bu^p, \ a > 0, \ b > 0, \ q > 0, \ p > 0$$

in [10].

Problem (1), (2) is equivalent to the integral equation in $C(\Omega)$

$$u(x) = \int_{\Omega} G(x,s) f(\lambda u(s)) ds, \qquad (3)$$

where: $G(x,s)$ is a Green's function for the operator $-\Delta u$ of the first boundary value problem in the domain

$$\Omega,\ x=(x_1,\ldots,x_n),\ s=(s_1,\ldots,s_n).$$

Now we rewrite equation (3) as follows:

$$u=Tu,$$

where: $Tu=\int\limits_{\Omega}G(x,s)f(\lambda u(s))ds$ is an operator with domain $D(T)=K$, K is a cone of non-negative functions in the space $C(\Omega)$.

It's naturally to expect that the existence and uniqueness of the positive solution of equation (1), and hence problem (1), (2), significantly depends on properties of the operator T and the form of $f(\lambda,u)$. The cases of monotone and antitone operator Tu are considered in [11-13].

Since the construction of Green's functions can be quite complicated even for two-dimensional problems there are only few cases for which a constructive solution can be obtained. In complex domains Green's quasifunction method can be used [14]. The method is based on the construction of a boundary equation. R-functions theory plays a significant role in solving this task [14, 15, 16].

In this work we investigate the following problem [8]

$$\begin{aligned} -\Delta u &= \lambda u^q + u^p \quad \forall x\in\Omega, \\ u>0, &\quad u|_{\partial\Omega}=0, \end{aligned} \tag{4}$$

where: $0<q<1<p,\quad \lambda>0$.

The equation of the problem (4) is a stationary heat conduction equation with two sources of different power and describes heat distribution over a plate (domain Ω) that doesn't change in time. It happens when stationary sources of heat act for a long time and transitional processes caused by them have been finished. The terms λu^q and u^p represent the power of heat sources.

EXISTENCE OF POSITIVE SOLUTIONS

Problem (4) is equivalent to the integral equation in $C(\Omega)$

$$u(x)=\int\limits_{\Omega}G(x,s)[\lambda u^q(s)+u^p(s)]ds. \tag{5}$$

We need following definitions in the sequel [17-19].

Definition 1. A convex closed set K in Banach space E is called a *cone* if this set contains, together with each element u, ($u\neq\theta$), all the elements of the form tu for $t\geq 0$ and does not contain the element $-u$, where θ is the zero element of E.

Definition 2. The cone K is called *normal* if there exists an $N(K)$ such that:

$$\|u\|\leq N(K)\|v\| \text{ for } 0\leq u\leq v,\ u,v\in K.$$

More precisely, the cone K is called *normal* if there exists a $\delta>0$ such that the inequality:

$$\|f_1+f_2\|\geq\delta$$

is satisfied for all

$$f_1, f_2\in K,\ \|f_1\|=\|f_2\|=1.$$

The cone of non-negative functions is normal in the space C.

Definition 3. The collection of elements $u\in K$ for which $v_0\leq u\leq w_0$ is called the *conical interval* $<v_0,w_0>$.

Definition 4. An operator T is *monotone* if $Tv\leq Tw$ follows from $v\leq w,\ v,w\in K$.

Definition 5. An operator T is *positive* if $TK\subset K$.

Definition 6. Let E and F be Banach spaces. An operator, acting from E into F, is called *completely continuous* if it maps every bounded set of the space E onto a (relatively) compact set of the space F.

Definition 7. Let $f(\lambda,x,u(x))$ be a non-negative and concave function (i.e.

$$f(\lambda,x,tu(x))-t\,f(\lambda,x,u(x))>0 \tag{6}$$

for all $t\in(0,1)$, $u>0$ and $x\in\Omega$). Then an operator

$$Tu=\int\limits_{\Omega}G(x,s)f(\lambda,s,u(s))ds$$

is called u_0-concave on K if

$$\alpha u_0(x)\leq Tu\leq\beta u_0(x)\ \ \forall u\geq 0,\ \alpha,\beta>0, \tag{7}$$

where: $u_0\in K$ is a fixed non-zero element.

Suppose K is a cone of non-negative functions in $C(\Omega)$. Let

$$u=Tu \tag{8}$$

be an operator equation defined over K, where:

$$Tu=\int\limits_{\Omega}G(x,s)[\lambda u^q(s)+u^p(s)]ds.$$

Since the cone K is normal and the function $f(u)=\lambda u^q+u^p$ is continuous in u, it follows that the operator T is completely continuous if it maps $C(\Omega)$ on itself [17, 18].

First, we state

Lemma 1. The operator T has following properties:

1) T is monotone.

2) There is a conical interval $< v_0, w_0 >$ such that

$$T < v_0, w_0 > \subset < v_0, w_0 >.$$

3) T is u_0-concave, where:

$$u_0 = \int_\Omega G(x,s)ds.$$

Proof. 1) The proof is trivial.

2) Let us build $< v_0, w_0 >$.

It is advised in [17] to put $v_0 = 0$ if $f(\lambda, x, u)$ is monotonically increasing in *u*. Following this advice we get

$$v_1 = Tv_0 = \int_\Omega G(x,s)[\lambda v_0^q + v_0^p]ds = 0.$$

Therefore, the interval's left endpoint stays still if we apply the successive approximation scheme:

$$u_{n+1}(x) = \int_\Omega G(x,s)[\lambda u_n^q(s) + u_n^p(s)]ds, \quad n = 1,2,\ldots. \tag{9}$$

It means that we obtain approximations from above only instead of two-sided ones.

Now we introduce the following concept. Let v_0 be

$$v_0(x) = \varepsilon\omega(x),$$

where: $\omega(x) > 0$ in Ω, $\omega(x)|_{\partial\Omega} = 0$, $\varepsilon = const > 0$.

Remark 1. The function $\omega(x)$ can be constructed practically for any domain using R-functions theory [14].

Hence,

$$v_1 = \int_\Omega G(x,s)[\lambda v_0^q + v_0^p]ds =$$
$$= \int_\Omega G(x,s)[\lambda\varepsilon^q\omega^q(s) + \varepsilon^p\omega^p(s)]ds =$$
$$= \int_\Omega G(x,s)\varepsilon^q[\lambda\omega^q(s) + \varepsilon^{p-q}\omega^p(s)]ds =$$
$$= \varepsilon^q \int_\Omega G(x,s)[\lambda\omega^q(s) + \varepsilon^{p-q}\omega^p(s)]ds.$$

From the inequality $v_1 \ge v_0$ it follows that

$$\varepsilon^q \int_\Omega G(x,s)[\lambda\omega^q(s) + \varepsilon^{p-q}\omega^p(s)]ds \ge \varepsilon\omega(x)$$

or

$$\int_\Omega G(x,s)[\lambda\omega^q(s) + \varepsilon^{p-q}\omega^p(s)]ds \ge$$
$$\ge \frac{\varepsilon\omega(x)}{\varepsilon^q} = \varepsilon^{1-q}\omega(x).$$

Then, squaring the last expression and applying the Cauchy-Schwarz inequality $|(u,v)| \le \|u\|\|v\|$:

$$\left(\int_\Omega G(x,s)[\lambda\omega^q(s) + \varepsilon^{p-q}\omega^p(s)]ds\right)^2 \ge$$
$$\ge \varepsilon^{2(1-q)}\omega^2(x),$$
$$\int_\Omega G^2(x,s)ds \int_\Omega [\lambda\omega^q(s) + \varepsilon^{p-q}\omega^p(s)]^2 ds \ge$$
$$\ge \varepsilon^{2(1-q)}\omega^2(x)$$

or

$$\omega^2(x) \le \varepsilon^{2(q-1)} M \int_\Omega [\lambda\omega^q(s) + \varepsilon^{p-q}\omega^p(s)]^2 ds,$$

where: $M = \max\limits_{x\in\Omega} \int_\Omega G^2(x,s)ds.$

Finally, we obtain

$$\max_{x\in\Omega} \omega^2(x) \le \le \varepsilon^{2(q-1)} M \int_\Omega [\lambda\omega^q(s) + \varepsilon^{p-q}\omega^p(s)]^2 ds \tag{10}$$

and this estimate is satisfied for any domain Ω.

Let us find w_0. First we put $w_0 = \beta = const > 0.$ Then, using the inequality $w_1 \le w_0$ and scheme (9) we obtain

$$w_1(x) = \int_\Omega G(x,s)[\lambda w_0^q(s) + w_0^p(s)]ds =$$
$$= \int_\Omega G(x,s)[\lambda\beta^q + \beta^p]ds \le \beta.$$

It now follows that

$$L \le \frac{\beta}{\lambda\beta^q + \beta^p}, \tag{11}$$

where: $L = \max\limits_{x\in\Omega} \int_\Omega G(x,s)ds.$

Thus, conditions (10) and (11) link parameters *p*, *q* and ε, β. The latter ones define the conical interval

$$< v_0 = \varepsilon\omega(x), w_0 = \beta >.$$

3) In order to show u_0-concavity we will use Definition 7.

Since we have shown how to build $< v_0, w_0 >$, it follows that (7) is satisfied.

Furthermore, from (6) $\lambda > \dfrac{\beta^{p-q}(t-t^p)}{t^q - t}$.

Now we define $g(t) := \dfrac{t-t^p}{t^q - t}$.

Solutions of the equation

$$(1-pt^{p-1})(t^q - t) - (qt^{q-1} - 1)(1-t^p) = 0 \quad (12)$$

define maximum values of the function $g(t)$ for $t \in (0,1)$.

Let $t_* \in (0,1)$ be a solution of (12). Now note that

$$\lim_{t \to 1} g(t) = \frac{p(p-1)}{q(q-1)}.$$

This implies that the parameters λ, q, p, and the constant β must also satisfy

$$\lambda > \max\left\{\beta^{p-q}\frac{p(p-1)}{q(q-1)}, \beta^{p-q}\frac{t_* - t_*^p}{t_*^q - t_*}\right\}. \quad (13)$$

This completes the proof of the lemma.

Now, we build an iteration process for equation (8) by the following scheme

$$\begin{aligned} v_{n+1}(x) &= \int_\Omega G(x,s)[\lambda v_n^q(s) + v_n^p(s)]ds, \\ &\qquad n = 0,1,\ldots, \\ w_{n+1}(x) &= \int_\Omega G(x,s)[\lambda w_n^q(s) + w_n^p(s)]ds, \\ &\qquad n = 0,1,\ldots. \end{aligned} \quad (14)$$

The main result of this paper is

Theorem 1. Process (14) converges to $u^*(x)$ from both sides with respect to the norm of space $C(\Omega)$ if λ, q, p, ε, β satisfy (10), (11), and (13), where $u^*(x)$ is an exact positive single solution of equation (5) and

$$v_0 < v_1 < \ldots < u^* < \ldots < w_1 < w_0.$$

Proof. First, we know that the cone $K \in C(\Omega)$ is normal. The operator T is completely continuous $\forall u \in K$, monotone and maps conical interval $< v_0, w_0 >$ into itself by Lemma 1 if λ, q, p, ε, β satisfy (10), (11), and (13). It now follows that the equation has exactly one positive solution [17].

Then since the operator T is also u_0-concave by Lemma 1 and the cone K is normal it follows that process (14) converges to $u^*(x)$ from both sides with respect to the norm of space $C(\Omega)$ [17].

This completes the proof of the theorem.

GREEN'S QUASIFUNCTION

Rvachev V.L. proposed to consider a special function which is close in particular sense to Green's one [14]. It's called Green's quasifunction. Now let's see how it can be established for problem (4).

Let $\psi = 0$ be the normalized boundary equation of the first order on boundary $\partial\Omega$, namely

$$\begin{aligned} \psi(x) = 0, \quad |\nabla\psi| = 1, \quad x \in \partial\Omega, \\ \psi(x) > 0, \quad x \in \Omega. \end{aligned} \quad (15)$$

Now we put

$$\zeta(x,s) = -\frac{1}{2}\ln(r^2 + 4\psi(x)\psi(s)), \quad \Omega \subset R^2,$$

$$\zeta(x,s) = \left[r^2 + 4\psi(x)\psi(s)\right]^{-\frac{1}{2}}, \quad \Omega \subset R^3,$$

where: $r = |x - s|$.

The Green's quasifunction can be established as follows

$$G_2(x,s) = \frac{1}{2\pi}\left[\ln\frac{1}{r} - \zeta(x,s)\right], \quad \Omega \subset R^2,$$

$$G_2(x,s) = \frac{1}{4\pi}\left[\frac{1}{r} - \zeta(x,s)\right], \quad \Omega \subset R^3.$$

Then problem (4) can be reduced to nonlinear integral equation

$$\begin{aligned} u(x) = \int_\Omega G_2(x,s)[\lambda u^q(s) + u^p(s)]ds + \\ + \int_\Omega u(s)K(x,s)ds, \end{aligned} \quad (16)$$

where: $K(x,s) = -\dfrac{1}{2\pi}\left(\dfrac{\partial^2}{\partial s_1^2} + \dfrac{\partial^2}{\partial s_2^2}\right)\zeta(x,s), \Omega \subset R^2,$

$$K(x,s) = -\frac{1}{4\pi}\left(\frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2} + \frac{\partial^2}{\partial s_3^2}\right)\zeta(x,s), \Omega \subset R^3.$$

Now we reduce (16) to a sequence of linear integral equations by applying the method of successive approximations [20]

$$u_{m+1}(x) - \int_\Omega u_{m+1}(s)K(x,s)ds =$$
$$= \int_\Omega G_2(x,s)[\lambda u_m^q(s) + u_m^p(s)]ds, \qquad (17)$$
$$m = 1,2,\ldots,$$

where: $u_1(x) = \delta\psi(x)$, $\delta = const > 0$.

Each of equations (17) can be solved by Bubnov-Galerkin method [21]. In that case we have the following sequence of the solution approximations

$$u_{m,k}(x) = \sum_{i=1}^{k} c_{m,i}\phi_i(x),$$

where: $\phi_i(x)$ is a coordinate sequence, $c_{m,i}$ ($i = \overline{1,k}$, $m = 2,3,\ldots$) is a solution of a system of linear algebraic equations:

$$\sum_{i=1}^{k} c_{2,i}\left[\int_\Omega \phi_i(x)\phi_j(x)dx - \right.$$
$$\left. - \int_\Omega\int_\Omega K(x,s)\phi_i(s)\phi_j(x)dsdx\right] =$$
$$= \int_\Omega\int_\Omega G_2(x,s)[\lambda u_1^q(s) + u_1^p(s)]\phi_j(x)dsdx,$$
$$j = \overline{1,k},$$
$$\sum_{i=1}^{k} c_{m,i}\left[\int_\Omega \phi_i(x)\phi_j(x)dx - \right. \qquad (18)$$
$$\left. - \int_\Omega\int_\Omega K(x,s)\phi_i(s)\phi_j(x)dsdx\right] =$$
$$= \int_\Omega\int_\Omega G_2(x,s)[\lambda u_{m-1,k}^q(s) +$$
$$+ u_{m-1,k}^p(s)]\phi_j(x)dsdx,$$
$$j = \overline{1,k}, \quad m = 3,4,\ldots.$$

COMPUTATIONAL EXPERIMENTS

As an illustration of process (14) we now look at the examples in two domains using both Green's functions and quasifunctions.

Example 1 (Unit Disk for Green's Function). Take

$$\Omega = \{x = (x_1, x_2) \mid 1 - x_1^2 - x_2^2 > 0\}. \qquad (19)$$

The corresponding Green's function in the domain Ω is:

$$G(x,s) = \frac{1}{2\pi}\left(\ln\frac{1}{|x-s|} - \ln\frac{1}{\rho|x-s^1|}\right),$$

where: $s \in \Omega$ is a fixed point, s^1 is an 'image point' on the prolonged line segment from the disk center *O* to *s* such that $\rho\rho^1 = 1$, ρ is a distance from *O* to *s*, ρ^1 is a distance from *O* to s^1 (see Fig. 1).

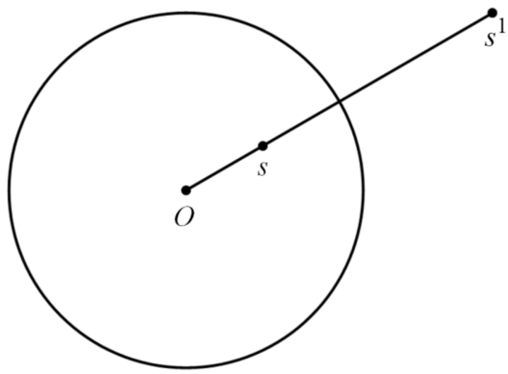

Fig. 1. Points for Green's function expression in Disk domain

The function $\omega(x)$ is:

$$\omega(x) = 1 - x_1^2 - x_2^2.$$

It means that:

$$\max_{x\in\Omega} \omega(x) = 1.$$

Now, by (10), so that:

$$1 < \frac{M\pi}{\varepsilon^{2(1-q)}}\left[\frac{\lambda^2}{2q+1} + \frac{2\lambda\varepsilon^{p-1}}{p+q+1} + \frac{\varepsilon^{2(p-q)}}{2p+1}\right]. \qquad (20)$$

We have $M \approx 0.04$, $L \approx 0.25$.

Using (11), (13), and (20) we put $p = 2$, $q = 0.5$, $\lambda = 3$, $\varepsilon = 0.5$, $\beta = 1.5$.

The surface of the upper approximation of the solution w_{18} and its level lines are illustrated in Fig. 2 and Fig. 3 respectively.

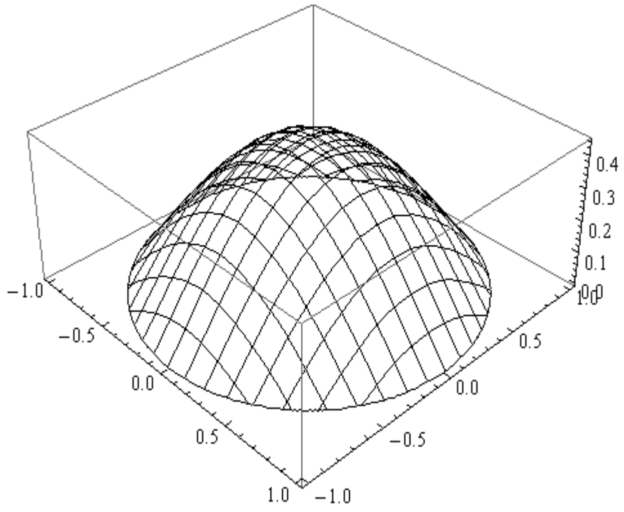

Fig. 2. The surface of w_{18}

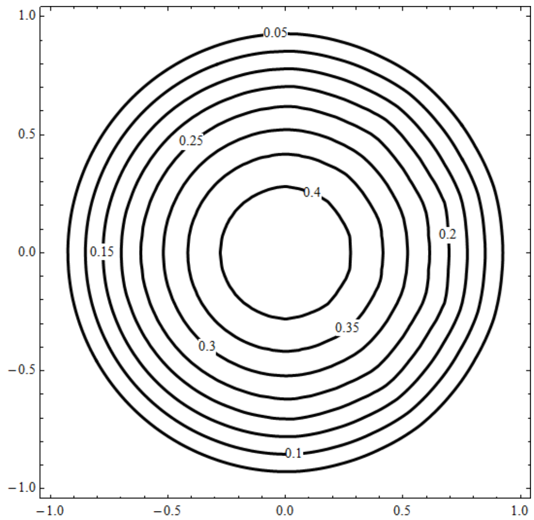

Fig. 3. The level lines of w_{18}

The values of the approximations v_{18} and w_{18} at the points of domain Ω in polar coordinates (ρ_i, ϕ_j), $\rho_i = 0.2i$, $\phi_j = 0.1 j\pi$, $i = \overline{1,4}$, $j = \overline{1,5}$ are shown in Table 1.

Table 1. The values of v_{18} and w_{18}

ϕ		ρ			
		0.2	0.4	0.6	0.8
$\frac{\pi}{10}$	w_{18}	0.419061	0.357569	0.258870	0.133876
	v_{18}	0.418969	0.357491	0.258815	0.133848
$\frac{\pi}{5}$	w_{18}	0.419073	0.357322	0.258291	0.133475
	v_{18}	0.418981	0.357244	0.258235	0.133447
$\frac{3\pi}{10}$	w_{18}	0.419096	0.357075	0.257844	0.133076
	v_{18}	0.419004	0.356997	0.257788	0.133048
$\frac{2\pi}{5}$	w_{18}	0.419131	0.357245	0.258149	0.133283
	v_{18}	0.419039	0.357167	0.258094	0.133255
$\frac{\pi}{2}$	w_{18}	0.419188	0.358912	0.261326	0.135752
	v_{18}	0.419095	0.358834	0.261270	0.135723

Example 2 (Unit Disk for Green's Quasifunction). Let

$$\psi(x) = \frac{1}{2}(1 - x_1^2 - x_2^2) \qquad (21)$$

be the normalized boundary equation of the first order on $\partial\Omega$, where Ω is defined by (19). Indeed, conditions (15) are satisfied for (21).

In this case equation (16) will be reduced to (5) which means that Greens' quasifunction is equal to Green's function in Disk region.
Therefore, the results will be the same as in Example 1.

Example 3 (Unit Half Disk for Green's Function). Take

$$\Omega = \left\{ x = (x_1, x_2) \,|\, 1 - x_1^2 - x_2^2 > 0, x_2 > 0 \right\} \qquad (22)$$

The corresponding Green's function in the domain Ω is

$$G(x,s) = \frac{1}{2\pi}\left(\ln\frac{1}{|x-s|} - \ln\frac{1}{\rho\,|x-s^1|} - \right.$$
$$\left. -\ln\frac{1}{|x-s'|} + \ln\frac{1}{\rho\,|x-s^{1'}|} \right),$$

where: *s*, s^1, ρ are described in Example 1, s', $s^{1'}$ are 'image points' corresponding to *s* and s^1 respectively (see Fig. 4).

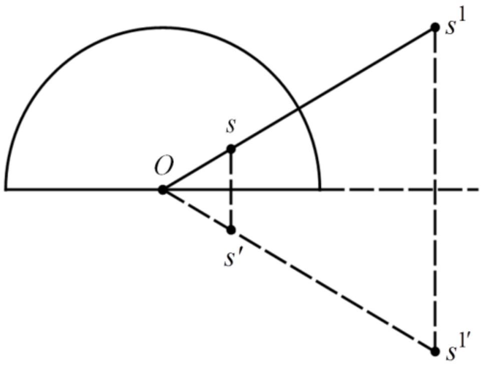

Fig. 4. Points for Green's function expression in Half Disk domain

The function $\omega(x)$ is $\omega(x) = x_2(1 - x_1^2 - x_2^2)$ meaning that $\max\limits_{x\in\Omega}\omega(x) = 1.$

In this case we have $M \approx 0.015,\ \ L \approx 0.097.$

Using (11), (13), and (20) we put $p = 2$, $q = 0.5$, $\lambda = 8$, $\varepsilon = 0.5$, $\beta = 1$.

The surface of the upper approximation of the solution w_{15} and its level lines are illustrated in Fig. 5 and Fig. 6 respectively.

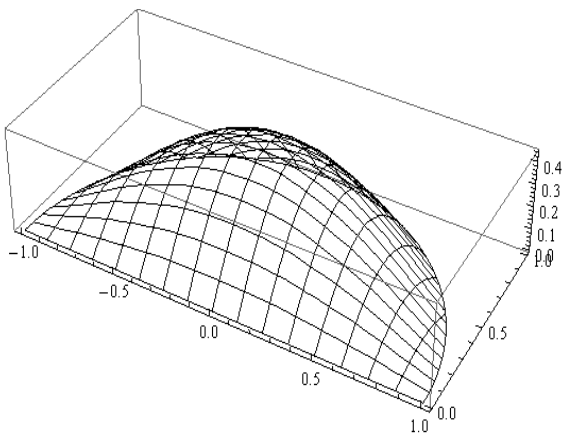

Fig. 5. The surface of w_{15}

The values of the approximations v_{15} and w_{15} at the points of domain Ω in polar coordinates (ρ_i, ϕ_j), $\rho_i = 0.2i$, $\phi_j = 0.1j\pi$, $i = \overline{1,4}$, $j = \overline{1,5}$ are shown in Table 2.

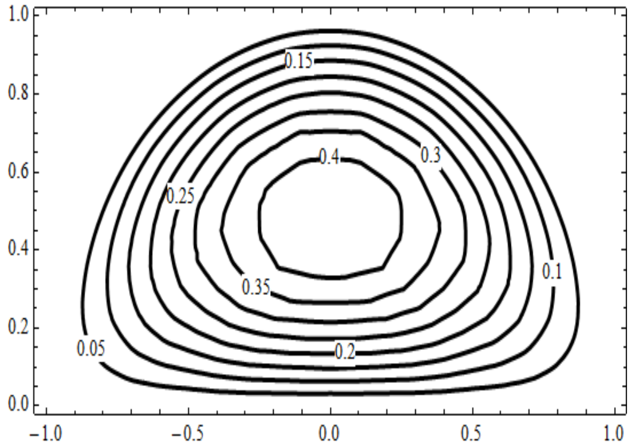

Fig. 6. The level lines of w_{15}

Example 4 (Unit Half Disk for Green's Quasifunction).

Let

$$\psi(x) = \frac{1}{2}(1 - x_1^2 - x_2^2)x_2 \qquad (23)$$

be the normalized boundary equation of the first order on $\partial\Omega$, where Ω is defined by (22). Indeed, conditions (15) are satisfied for (23).

Now we put $\delta = 0.5$, so that

$$u_0(x) = \frac{1}{4}(1 - x_1^2 - x_2^2)x_2.$$

Then we select the following coordinate sequence

$$\phi_i(x) = \psi(x)P_{i_1}(x_1)P_{i_2}(2x_2 - 1),$$
$$i_1 = \overline{0,2},\quad i_2 = \overline{0,2-i_1},$$

where: $i = \overline{1,k}$, $k = 6$, $P_m(z)$ are Legendre polynomials

$$P_m(z) = \frac{1}{2^m m!}\frac{d^m}{dz^m}\left[(z^2-1)^m\right].$$

Solving the system of equations (18), we get the approximation of the solution $u_{14,6}$. Its surface and level lines are illustrated in Fig. 7 and Fig. 8 respectively.

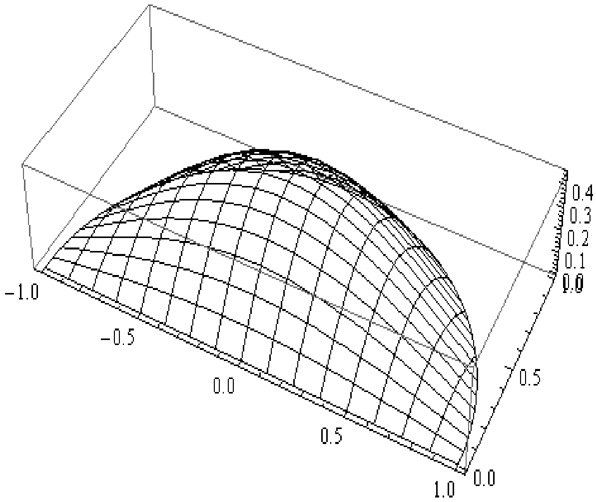

Fig. 7. The surface of $u_{14,6}$

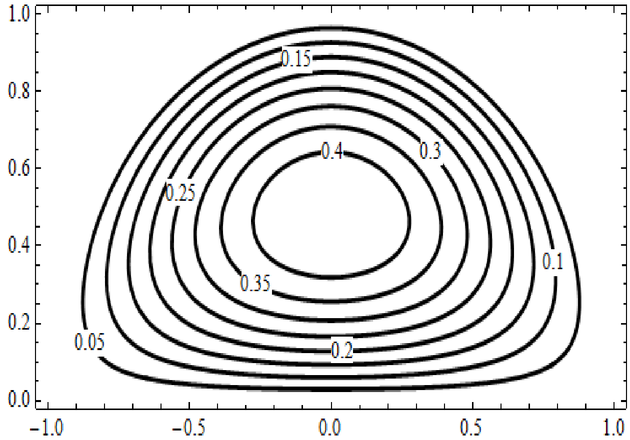

Fig. 8. The level lines of $u_{14,6}$

Table 2. The values of v_{15} and w_{15}

ϕ		ρ			
		0.2	0.4	0.6	0.8
$\frac{\pi}{10}$	w_{15}	0.094379	0.159946	0.167969	0.109479
	v_{15}	0.094362	0.159917	0.167939	0.109459
$\frac{\pi}{5}$	w_{15}	0.175253	0.284471	0.286695	0.179712
	v_{15}	0.175221	0.284419	0.286643	0.179680
$\frac{3\pi}{10}$	w_{15}	0.235852	0.371114	0.363978	0.223259
	v_{15}	0.235808	0.371045	0.363910	0.223218
$\frac{2\pi}{5}$	w_{15}	0.273067	0.421881	0.407777	0.247385
	v_{15}	0.273017	0.421802	0.407701	0.247339
$\frac{\pi}{2}$	w_{15}	0.285590	0.438515	0.421797	0.255014
	v_{15}	0.285537	0.438432	0.421718	0.254967

The values of the approximation $u_{14,6}$ at the points of domain Ω in polar coordinates (ρ_i, ϕ_j), $\rho_i = 0.2i$, $\phi_j = 0.1 j\pi$, $i = \overline{1,4}$, $j = \overline{1,5}$ are shown in Table 3.

Table 3. The values of $u_{14,6}$

ϕ	ρ			
	0.2	0.4	0.6	0.8
$\frac{\pi}{10}$	0.097834	0.160302	0.167488	0.111392
$\frac{\pi}{5}$	0.180005	0.285107	0.287184	0.183040
$\frac{3\pi}{10}$	0.241369	0.372598	0.365651	0.226659
$\frac{2\pi}{5}$	0.279068	0.424053	0.410050	0.250792
$\frac{\pi}{2}$	0.291761	0.441007	0.424467	0.258674

CONCLUSIONS

We have built an iteration process that converges to a positive solution of (4) from both sides. Also, we have introduced a new approach for constructing conical intervals, as a left endpoint we $v_0(x) = \varepsilon\omega(x)$ instead of $v_0(x) = 0$, where: $\omega(x) > 0$ in Ω, $\omega(x)|_{\partial\Omega} = 0$, and $\varepsilon = const$.

This approach can be used when the lower approximations don't move from the starting position.

We've obtained a condition that links parameters λ, q, p, ε, β and guarantees existence and uniqueness of a positive solution.

By building the cone segment $\langle v_0, w_0 \rangle$ we provide an a priori estimate of the solution, since $v_0 \le u \le w_0$. The actual two-sided approximations allow us to make a posteriori conclusions.

The algorithm implementation simplicity and relatively small computational resources are the main advantages of the provided method.

Green's quasifunction method has been investigated to compare the results. The functions $\omega(x)$ and $\psi(x)$ can be constructed using R-functions theory [14] in case of domains with complex boundary.

The experimental results in unit disk and unit half disk have shown the efficiency of the provided method. It can be used to solve boundary value problems for stationary heat conduction equations of the form (4) or other problems that are reduced to (4).

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