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# ELEMENTS TRANSPOSITIONS AND THEIR IMPACT ON THE CYCLIC STRUCTURE OF PERMUTATIONS

*I. Grebennik, O.Chorna*

*Kharkiv National University of Radio Electronics,*
*e-mail: igorgrebennik@gmail.com*

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Abstract**.** The objective of this paper is the investigation of the cyclic structure and permutation properties based on neighbor elements transposition properties and the properties of the permutation polyhedron. In this paper we consider special type of transpositions of elements in a permutation. A feature of these transpositions is that they corresponding to the adjacency criterion in a permutation polyhedron. We will investigate permutation properties with the help of the permutation polyhedron by using the immersing in the Euclidian space. Six permutation types are considered in correspondence with the location of arbitrary components. We consider the impact of the corresponding components on the cyclic structure of permutations depending on the type of a permutation. In this paper we formulate the assertion about the features of the impact of transpositions corresponding to the adjacency criterion on the permutations consisting of the one cycle. During the proof of statement all six types of permutations are considered and clearly demonstrated that only two types arrangement of the elements in the cycle contribute to the persistence a single cycle in the permutation after the impact of two transpositions. Research conducted in the given work, will be further employed in mathematical modeling and computational methods. Especially for solving combinatorial optimization problems and for the generation of combinatorial objects with a predetermined cyclic structure.

Key words. Permutations set, permutation polyhedron, adjacency criterion, permutation properties, transposition, combinatorics.

## INTRODUCTION

This research is devoted to the investigation of two consecutive transpositions of neighboring by value generative elements of a permutation and their impact on the cyclic structure of the permutation being considered, and also their relative location on the permutation polyhedron.

In this paper we introduce a classification of permutations in dependence of some components relative location and the impact of these components transpositions on the permutation structure.

The aim of this paper is the investigation of the cyclic structure and permutation properties based on neighbor elements transposition properties and the properties of the permutation polyhedron.

## THE ANALYSIS OF RECENT RESEARCHES AND PUBLICATIONS

Permutations sets are very often considered in theoretical and applied research in the field of combinatorics and combinatorial optimization [1–18]. By now many properties of permutations have been investigated, in particular those associated with the cyclic structure of permutations. Some methods and algorithms allowing the representation of permutations as the product of cycles and the generation of permutations having a predefined cyclic structure are known [1, 4, 6-9, 10-14].

A well-known way for investigating combinatorial sets is their immersion into the Euclidian space, which allows using tools of continuous mathematics when analyzing combinatorial problems [2, 5]. The convex hull of a permutations set immersed into the Euclidian space is a permutation polyhedron [3, 16]. One of the basic properties of this polyhedron is the fact that the adjacency criterion for its vertexes is satisfied [1-3].

The geometric and analytic interpretations of one transposition corresponding to the permutation adjacency criterion are well known and were investigated earlier [2, 3, 18, 19]. The same issues discussed in this paper are used in the mathematical modeling and computational methods to describe and solve many economic, social and applied problems [20-23].

## BASIC DEFINITIONS

Let $P_n^C$ be the set of cyclic permutations without repetition from $n$ real numbers [4, 9]:

$$a_1 < a_2 < ... < a_n: \ \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in P_n^C ,$$

where:

$$c_i \in R , \ i \in J_n = \{1, 2, ..., n\} .$$

Consider the notion of permutation in detail [9].

*Definition 1.* A linear ordering of the elements from a certain generating set $A = \{a_1, a_2, ..., a_n\}$ is called a permutation:

$$\pi = \pi(a_1, a_2, ..., a_n) = (\pi(a_1), \pi(a_2), ..., \pi(a_n)) = (a_{i_1}, a_{i_2}, ..., a_{i_n})$$

or, if it is necessary to stress the fact that it contains $n$ elements, $n$-permutation.

We denote as $P_n$ the set of permutations generated by the elements $a_1 < a_2 < ... < a_n$.

Consider a certain permutation $\pi = (\pi(a_1), \pi(a_2), ..., \pi(a_n)) \in P_n$ and its element $\pi(a_i) = a_j$, $\forall i, j \in J_n$. Then we can write down:

$$\pi(a_j) = \pi(\pi(a_i)) = \pi^2(a_i) .$$

Generally, this formula can be represented in the following form:

$$\pi^{k-1}(a_j) = \pi(\pi^{k-1}(a_i)) = \pi^k(a_i) , \quad \forall i, j \in J_n , \ k \le n .$$

Thus if for some $l \ge 1$ we have:

$$\pi^l(a_i) = a_i , \ i \in J_n$$

and all the elements

$$a_i, \pi(a_i), \pi^2(a_i), ..., \pi^{l-1}(a_i)$$

are different, the sequence:

$$(a_i, \pi(a_i), \pi^2(a_i), ..., \pi^{l-1}(a_i))$$

is called [2] an $l$ length cycle.

*Definition 2*. A cyclic permutation is such a permutation $\pi$ from $n$ elements that contains a single $n$ length cycle [2], i.e.:

$$\pi^n(a_i) = a_i , \ \forall i \in J_n .$$

We denote such permutations as $\pi_C$. Note that, according to [4]:

$$Card \ P_n^C = (n-1)! .$$

Let us consider a cyclic permutation example. In this work we will use the following notation way. We record two rows so that the generating elements are written not in the increasing order but in the order of their appearance in the cycle:

$$\begin{pmatrix} 1 & & 4 & & 6 & & 2 & & 5 & & 8 & & 3 & & 7 \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ 4 & & 6 & & 2 & & 5 & & 8 & & 3 & & 7 & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 6 & 2 & 5 & 8 & 3 & 7 \\ 4 & 6 & 2 & 5 & 8 & 3 & 7 & 1 \end{pmatrix} .$$

One of the widespread directions of combinatorial research is the immersion (enclosure mapping) of a combinatorial set into the Euclidian space [5].

The immersion of combinatorial sets in the Euclidian space allows constructing combinatorial polyhedrons [1] with the help of which it is possible to investigate the properties of permutations sets classes in the Euclidian space.

Let us fulfill the enclosure mapping of the permutations set $P_n$ and cyclic permutations $P_n^C$ to the arithmetic Euclidian space $R^n$. According to [2, 5] the given mapping (which is called immersion) can be represented in the following form:

$$f : P \to R^n , \ \forall p = (p_1, p_2, ..., p_n) \in P ,$$
$$x = f(p) = (x_1, x_2, ..., x_n) \in E \subset R^n ,$$
$$x_i = p_i , \ i \in J_n .$$

As a result of the immersion $f$ we have one-to-one correspondence between each set:

$$P_n , \ P_n^C \text{ and } E \subset R^n :$$
$$E_n = f(P_n) , \ E_n^C = f(P_n^C) .$$

Because of the above operations it becomes possible to investigate properties of the elements of the subset $P_n^C \subset P_n$ with the help of the polyhedron $\varPi_n$.

## OBJECTIVES

To formulate goals consider relative location of some permutation components.

Consider a permutation polyhedron $\varPi_n$ generated by a set:

$$a_1 < a_2 < ... < a_n , \quad vert \ \varPi_n = E_n$$

is the set of its vertexes.

Since any cyclic permutation belongs to the set of permutations $P_n$:

$$\pi_C = (\pi(a_1), \pi(a_2), ..., \pi(a_n)) \in P_n$$

all cyclic permutations are vertexes of the permutation polyhedron $\varPi_n$.

Let us introduce:

$$\varPi_n^C \subset vert \ \varPi_n ,$$

which is the subset of permutation polyhedron vertexes corresponding to all possible cyclic permutations with the cycle of $n$ elements generated by the set:

$$a_1 < a_2 < ... < a_n , \quad \varPi_n^C = E_n^C .$$

Let us denote $v^C$ the vertex $v \in vert\ \Pi_n$ corresponding to a certain cyclic permutation $p \in P_n^C$ i.e. $v^C \in \Pi_n^C \subset vert\ \Pi_n$.

The adjacency criterion for the vertexes of the permutation polyhedron deals with the elements of the generating set $A = \{a_1, a_2, ..., a_n\}$ and their location in the permutation, i. e.: a vertex adjacent to the vertex $v = (a_{v_1}, a_{v_2}, a_{v_3}, ..., a_{v_n})$ that corresponds to the permutation $p \in P_n^C$ is any vertex corresponding to the permutation $p_k$ obtained from $p$ by the transposition of components equal to $k$ and $k+1$, $\forall k \in J_{n-1}$. And the two permutations $p_1, p_2 \in P_n$, corresponding to the vertexes $v_1, v_2 \in \Pi_n$ are called adjacent permutations if the vertexes $v_1, v_2$ are adjacent vertexes of the polyhedron $\Pi_n$ [19].

Further in this paper, without losing generality we suppose:

$$A = \{a_1, a_2, ..., a_n\} = \{1, 2, \ldots, n\}.$$

Let us consider the location of the components equal to $i, i+1$ and $j, j+1$, $i, j \in J_{n-1}, j \neq i$ in an arbitrary cyclic permutation $p \in P_n^C$ and write them down in the form of chains. The components can be located in a chain in six ways that we call types. Let us fix the first two elements of the chain. This is a certain component $x \in A = \{a_1, a_2, ..., a_n\}$ the value of which is arbitrary and the component $i$ such that $\pi(x) = i$. Thus the beginning of a chain always looks as follows:

$$(x, i, \_, \_, \_),$$

and further there are three positions in which the components $i+1, j, j+1$ can be located in a different order. It is this order that sets the type of a cyclic permutation for components $i, j$. The number of ordering ways for 3 components is equal to $3! = 6$, which is the number of permutations from 3 elements.

Let us write down all the six permutations from the elements $i+1, j, j+1$:

1) $j, i+1, j+1$;
2) $j+1, i+1, j$;
3) $i+1, j, j+1$;
4) $j+1, j, i+1$;
5) $i+1, j+1, j$;
6) $j, j+1, i+1$.

Consider the elements chains corresponding to the given sequences.

Since we consider the location of components in an arbitrary cyclic permutation $p \in P_n^C$ it is not important which component will be the beginning of a chain because in a cyclic permutation any component can be obtained from any other component by the number of steps $\leq n-1$. Further, in this paper, without losing generality, we will always start chains with a fixed component $i$ and an arbitrary component $x$ the value of which is inessential but the component $i$ is a mapping of $x$.

Thus using the above considerations let us show all the six location types for the components

$$i, i+1 \text{ and } j, j+1,\ i, j \in J_{n-1}, j \neq i$$

in an arbitrary cyclic permutation $p \in P_n^C$ (for all types $a, b, c \in \{0, ..., n-2\}$):

Type I:
$$\begin{pmatrix} x & & \pi^a(i) & & \pi^b(j) & & \pi^c(i+1) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ i & & j & & i+1 & & j+1 \end{pmatrix},$$

Type II:
$$\begin{pmatrix} x & & \pi^a(i) & & \pi^b(j+1) & & \pi^c(i+1) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ i & & j+1 & & i+1 & & j \end{pmatrix},$$

Type III:
$$\begin{pmatrix} x & & \pi^a(i) & & \pi^b(i+1) & & \pi^c(j) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ i & & i+1 & & j & & j+1 \end{pmatrix},$$

Type IV:
$$\begin{pmatrix} x & & \pi^a(i) & & \pi^b(j+1) & & \pi^c(j) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ i & & j+1 & & j & & i+1 \end{pmatrix},$$

Type V:
$$\begin{pmatrix} x & & \pi^a(i) & & \pi^b(i+1) & & \pi^c(j+1) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ i & & i+1 & & j+1 & & j \end{pmatrix},$$

Type VI:
$$\begin{pmatrix} x & & \pi^a(i) & & \pi^b(j) & & \pi^c(j+1) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ i & & j & & j+1 & & i+1 \end{pmatrix},$$

We will use the introduced definitions and types of cyclic permutations for the formalization and investigation of the adjacency properties for elements from the set $\Pi_n^C$.

## THE MAIN RESULTS OF THE RESEARCH

In correspondence with the criterion of vertexes adjacency in the permutation polyhedron $\Pi_n$ [1] for any vertex $v \in vert\ \Pi_n$ there are $(n-1)$ adjacent vertexes obtained from the $v$ transposition of the components equal to $i$ and $i+1$ correspondingly, $i \in J_{n-1}$. Note that this is true also for any vertex $v \in \Pi_n^C$.

For any vertex $v \in vert\, \Pi_n$ we will call the transposition of components equal to $i$ and $i+1$, $i \in J_{n-1}$ belonging to the same $k$ length cycle of the corresponding permutation $p \in P_n$ a "break" transposition, since as a result of this operation the vertex $v_1 \in vert\, \Pi_n$ adjacent to the original one will be obtained and the permutation $p_1 \in P_n$ corresponding to the obtained vertex contains at least two cycles of length $k_1$ and $k_2$, $k_1 + k_2 = k$.

Therefore for the vertexes $v \in vert\, \Pi_n$ one transposition of components equal to $i$ and $i+1$, $i \in J_{n-1}$ can be either "break" or "conjunction" of cycles to which these components belong. If the original vertex $v \in vert\, \Pi_n$ corresponds to the permutation $p \in P_n^C$ belonging to the set of cyclic permutations and has a single cycle of length $n$, any transposition of the components $i$ and $i+1$, $i \in J_{n-1}$ will be a "break" [18, 19].

Next let us consider if it is possible to keep the cyclicity of a permutation when we have two transpositions of components equal to

$$i, i+1 \text{ and } j, j+1,\ i, j \in J_{n-1}, j \neq i.$$

Consider the case where there are four components involved in two transpositions of components.

**Statement 1**. If in a certain permutation $p \in P_n^C$ 2 consecutive transpositions of the elements

$$i, i+1 \quad \text{and } j, j+1,\ i, j \in J_{n-1}, j \neq i, i+1$$

have been fulfilled then the obtained permutation $p_{i,j} \in P_n$ will be cyclic if the original permutation $p \in P_n^C$ for the given components $i, j$ belongs to the type I or II.

**Proof**. Let us consider the fulfillment of transpositions for all the six types of permutations for some components $i, i+1$ and $j, j+1$, $i, j \in J_{n-1}, j \neq i, i+1$ and demonstrate which types keep the property of cyclicity. For all the six types: $a, b, c \in \{0, ..., n-2\}$.

Type I: $j, i+1, j+1$. The original chain looks as follows:

$$\begin{pmatrix} x & \pi^a(i) & \pi^b(j) & \pi^c(i+1) \\ \downarrow\nearrow & \downarrow\ \nearrow & \downarrow\ \nearrow & \downarrow \\ i & j & i+1 & j+1 \end{pmatrix}.$$

Fulfill consecutively two transpositions: $i \leftrightarrow i+1$, $j \leftrightarrow j+1$ and represent the chain elements without changing the links to facilitate visual perception. As a result, we get:

$$\begin{pmatrix} x & \pi^a(i) & \pi^b(j) & \pi^c(i+1) \\ \downarrow\nearrow & \downarrow\ \nearrow & \downarrow\ \nearrow & \downarrow \\ i & j & i+1 & j+1 \end{pmatrix} \underset{\substack{i \leftrightarrow i+1 \\ j \leftrightarrow j+1}}{\Longrightarrow} \begin{pmatrix} x & \pi^c(i+1) & \pi^b(j) & \pi^a(i) \\ \downarrow\ \nearrow & \downarrow\ \nearrow & \downarrow\ \nearrow & \downarrow \\ i+1 & j & i & j+1 \end{pmatrix}.$$

Type II: $j+1, i+1, j$. The original chain looks as follows:

$$\begin{pmatrix} x & \pi^a(i) & \pi^b(j+1) & \pi^c(i+1) \\ \downarrow\nearrow & \downarrow\ \nearrow & \downarrow\ \nearrow & \downarrow \\ i & j+1 & i+1 & j \end{pmatrix}.$$

Fulfill consecutively two transpositions: $i \leftrightarrow i+1$, $j \leftrightarrow j+1$ and change places of chain elements without changing the links. We will get:

$$\begin{pmatrix} x & \pi^a(i) & \pi^b(j+1) & \pi^c(i+1) \\ \downarrow\nearrow & \downarrow\ \nearrow & \downarrow\ \nearrow & \downarrow \\ i & j+1 & i+1 & j \end{pmatrix} \underset{\substack{i \leftrightarrow i+1 \\ j \leftrightarrow j+1}}{\Longrightarrow} \begin{pmatrix} x & \pi^c(i+1) & \pi^b(j+1) & \pi^a(i) \\ \downarrow\ \nearrow & \downarrow\ \nearrow & \downarrow\ \nearrow & \downarrow \\ i+1 & j+1 & i & j \end{pmatrix}.$$

Type III: $i+1, j, j+1$. The original chain looks as follows:

$$\begin{pmatrix} x & \pi^a(i) & \pi^b(i+1) & \pi^c(j) \\ \downarrow\nearrow & \downarrow\ \nearrow & \downarrow\ \nearrow & \downarrow \\ i & i+1 & j & j+1 \end{pmatrix}.$$

Let us fulfill consecutively two transpositions: $i \leftrightarrow i+1$, $j \leftrightarrow j+1$ and change places for the chain elements without changing the links. We will get:

$$\begin{pmatrix} x & \pi^a(i) & \pi^b(i+1) & \pi^c(j) \\ \downarrow\nearrow & \downarrow\ \nearrow & \downarrow\ \nearrow & \downarrow \\ i & i+1 & j & j+1 \end{pmatrix} \underset{\substack{i \leftrightarrow i+1 \\ j \leftrightarrow j+1}}{\Longrightarrow} \begin{pmatrix} x & \pi^b(i+1) & \pi^a(i) & \pi^c(j) \\ \downarrow\ \nearrow & \downarrow & \updownarrow & \updownarrow \\ i+1 & j+1 & i & j \end{pmatrix}.$$

Type IV: $j+1, j, i+1$. The original chain looks as follows:

$$\begin{pmatrix} x & \pi^a(i) & \pi^b(j+1) & \pi^c(j) \\ \downarrow\nearrow & \downarrow\ \nearrow & \downarrow\ \nearrow & \downarrow \\ i & j+1 & j & i+1 \end{pmatrix}.$$

Fulfill consecutively two transpositions: $i \leftrightarrow i+1$, $j \leftrightarrow j+1$. We will get:

$$\begin{pmatrix} x & & \pi^a(i) & & \pi^b(j+1) & & \pi^c(j) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ i & & j+1 & & j & & i+1 \end{pmatrix} \underset{\substack{i \leftrightarrow i+1 \\ j \leftrightarrow j+1}}{\Longrightarrow} \begin{pmatrix} x & \pi^a(i) & & \pi^c(j)\pi^b(j+1) & \\ \downarrow & \downarrow & \nearrow\nwarrow & \downarrow & \updownarrow \\ i+1 & j & & i & j+1 \end{pmatrix}.$$

Type V: $i+1, j+1, j$. The original chain looks as follows:

$$\begin{pmatrix} x & & \pi^a(i) & & \pi^b(i+1) & & \pi^c(j+1) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ i & & i+1 & & j+1 & & j \end{pmatrix}.$$

Fulfill consecutively two transpositions: $i \leftrightarrow i+1$, $j \leftrightarrow j+1$. We will get:

$$\begin{pmatrix} x & & \pi^a(i) & & \pi^b(i+1) & & \pi^c(j+1) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ i & & i+1 & & j+1 & & j \end{pmatrix} \underset{\substack{i \leftrightarrow i+1 \\ j \leftrightarrow j+1}}{\Longrightarrow} \begin{pmatrix} x & & \pi^b(i+1)\pi^a(i)\pi^c(j+1) & & \\ \downarrow & \nearrow & \downarrow & \updownarrow & \updownarrow \\ i+1 & & j & i & j+1 \end{pmatrix}.$$

Type VI: $j, j+1, i+1$. The original chain looks as follows:

$$\begin{pmatrix} x & & \pi^a(i) & & \pi^b(j) & & \pi^c(j+1) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ i & & j & & j+1 & & i+1 \end{pmatrix}.$$

Fulfill consecutively two transpositions: $i \leftrightarrow i+1$, $j \leftrightarrow j+1$. We will get:

$$\begin{pmatrix} x & & \pi^a(i) & & \pi^b(j) & & \pi^c(j+1) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ i & & j & & j+1 & & i+1 \end{pmatrix} \underset{\substack{i \leftrightarrow i+1 \\ j \leftrightarrow j+1}}{\Longrightarrow} \begin{pmatrix} x & \pi^a(i) & & \pi^c(j+1)\pi^b(j) & \\ \downarrow & \downarrow & \nearrow\nwarrow & \downarrow & \updownarrow \\ i+1 & j+1 & & i & j \end{pmatrix}.$$

Thus, after fulfilling two transpositions of components $i, i+1$ and $j, j+1$, $i, j \in J_{n-1}, j \neq i$, in the permutation $p \in P_n^C$ only two types of the components original location in the chain correspond to keeping the cyclicity property. The other four types do not keep cyclicity after component transpositions and lead to appearing cycles of a length less than $n$:

$$\pi^k(z) = z,\ k < n,\ z \in \{i, i+1, j, j+1\}.$$

The statement is proven.

## CONCLUSIONS

1. The given work has been devoted to the investigation of adjacent permutations and their cyclic properties. We have investigated permutation properties with the help of the permutation polyhedron by using the immersing in the Euclidian space.
2. Based on the known adjacency criterion for the vertexes of the permutation polyhedron $\Pi_n$, similar transpositions of components in permutations have been investigated.
3. Depending on the location of arbitrary components $i, i+1$ and $j, j+1$, $i, j \in J_{n-1}, j \neq i, i+1$ six types of permutations have been introduced.
4. For these types the changes in the cyclic structure of a permutation that appear after fulfilling two consecutive transpositions of the corresponding components have been investigated.

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