Integer Linear Programming Models for the Problem of Covering a Polygonal Region by Rectangles

G. Scheithauer, Yu. Stoyan and T. Romanova

Abstract– The aim of the paper is to develop integer linear programming (ILP) models for the problem of covering a polygonal region by rectangles. We formulate a Beasley-type model in which the number of variables depends on the size parameters. Another ILP model is proposed which has $O(n^2 \max\{m, n\})$ variables where m is the number of edges of the target set and n is the number of given rectangles. In particular we consider the case where the polygonal region is convex. Extensions are also discussed where we allow the polygonal region to be a union of a finite number of convex subsets.

Index terms – Covering, Integer Linear Programming, Mathematical Modelling, Optimization

Introduction and Problem Formulation

In this paper the problem of covering a polygonal target set Ω by a finite number of given rectangles is considered. The main aim is to formulate integer linear programming or optimization models. Rotation of rectangles is not allowed. In particular, the target set is assumed to be an arbitrary convex polygon. Since covering with axes-parallel rectangles is considered the target set can also be assumed to be an orthogonally convex rectilinear polygon.

The decision version of problem CPR (Covering a Polygonal set with Rectangles) asks whether there exists a covering or not. It is known to be NP-complete since the decision version of the Bin Packing Problem ([6]) can be reduced to the CPR problem (cf. [4]). Notice, in difference to e.g. [4] where a finite number of points has to be covered, we consider the covering of an infinite point set. Therefore, the verification that a certain configuration of the rectangles forms a cover of the target set cannot be done by inspecting a finite number of points, another technique is needed.

A solution approach based on a so-called Γ-function is proposed in [9]. The Γ-function of a certain configuration of all given rectangles attains a non-negative value if and only if this configuration forms a feasible covering of the target set. Based on an enumeration scheme, instances for which no cover exists can require a lot of computational effort to prove that circumstance.

In [8] one-dimensional bar relaxations for the CPR problem are proposed to be used as necessary conditions for existence of coverings.

Covering problems are of interest in many fields of application. There are many relations between covering and packing or cutting problems. For an annotated survey on *Cutting and Packing* we refer to [5]. Covering problems arise naturally in a variety of applications. For a comprehensive overview we refer to [4].

As an example, query optimization in spatial databases is a source of covering problems. In this setting a query may correspond to a geometric region and be phrased in a generic form using geometric parameters. Given a set of existing geometric, parametrized, query regions and a set of points or regions, we might want to ask if there are values of the parameters that allow the query regions to cover the set of points or even regions. Another field of application (also mentioned in [4]) is shape recognition for robotics, graphics or image processing applications. In these cases it is sometimes useful to represent a shape as a collection of parts. However, given a collection of parts and a shape, it can be difficult to determine if the shape can be described by that collection of parts. If the goal is to obtain an outer approximation of the shape using the parts, then this can be posed as a covering problem.

Besides the decision problem whether a covering exists or not, related problems can be of interest. For instance, one can ask for a minimum number of (identical) rectangles needed to form a cover for the target region. Or, if there exist several covers one can look for a best cover where best means that some objective function is regarded.

In computational geometry, decomposition of a polygon is of high interest involving partitioning and covering problems. For details and further literature we refer to [10, 1, 7].

In the approach proposed in [9] to solve the problem of covering a compact polygonal region with a finite family of rectangles, the choice of a suitable starting configuration is in particular an essential aspect.

Therefore, and since the CPR problem is hard to solve, enumeration algorithms like branch-and-bound have to be used in general in an exact solution approach. Another possibility to attack the CPR problem consists in formulating integer linear programming models and to solve them.

The aim of this contribution is to develop models and to discuss advantages and disadvantages with respect to the construction of exact solution approaches. Without loss of generality, we assume that all input-data are integers.

The paper is organized as follows. In the rest of this section we give the input-parameter and state the problem considered in the paper. In the next section a Beasley-type model of the covering problem is discussed. Then, in section 3, a basic model is developed. In section 4, we present a corresponding ILP model. Some extensions and alternative formulations are discussed in section 5. Finally, some conclusions follow.

Within this paper we consider the following optimization problem. We assume that target set Ω is a convex polygon. Consequently, we can use the representation

$$\begin{aligned} \Omega &= \{(x,y) : g_j(x,y) \le 0,\ j \in I_\Omega\} \\ &= \operatorname{conv}\{(X_j, Y_j) : j \in I_\Omega\}, \\ I_\Omega &= \{1, \dots, m\}. \end{aligned}$$

All the linear functions g_j, $j \in I_\Omega$ are assumed to be necessary for the representation of Ω. Hence, the number of vertexes (corners) (X_j, Y_j) of Ω coincides with the (minimum) number of functions g_j. In order to cover the given target set Ω the following rectangles are available:

$$\begin{aligned} R_i &= \{(x,y) : -a_i \le x \le a_i, -b_i \le y \le b_i\}, \\ & i \in I_n = \{1, \dots, n\} \end{aligned}$$

where $2a_i$ is the length and $2b_i$ the width of rectangle R_i. We assign to each rectangle its (positive) value c_i, $i \in I_n$, e. g. $c_i = a_i b_i$, and we denote the placement parameters (translation vectors) by $u_i = (x_i, y_i)$, $i \in I_n$. Hence,

$$\begin{aligned} R_i(x_i, y_i) &= \{(x,y) : x_i - a_i \le x \le x_i + a_i, \\ & \qquad y_i - b_i \le y \le y_i + b_i\} \end{aligned}$$

represents the translated rectangle R_i. Then the problem CPR under consideration is:

> Find a subset I^* of I_n of the rectangles and corresponding placement parameters u_i, $i \in I^*$ such that $\cup_{i \in I^*} R_i(u_i)$ forms a cover of Ω and its total valuation $\sum_{i \in I^*} c_i$ is minimal.

We always assume that problem CPR has a solution. This can be done by adding some sufficiently large rectangle R_0 (i. e. the minimum Ω enclosing rectangle) with $c_0 > \sum_{i \in I_n} c_i$. Consequently, if the optimal value of the CPR problem is not smaller than c_0 then the original problem has no covering.

A Beasley-type Model: ILP Model 1

Beasley [3, 2] proposed an integer linear programming (ILP) model with 0/1-variables for the two-dimensional rectangle packing problem. There, the 0/1-variables x_{ipq} are used to describe the placement of the reference point (lower left corner) of rectangle R_i at position (p, q).

For an ILP model of the CPR problem we can also use these x_{ipq}-variables. Let

$$\begin{aligned} X_E &:= \max\nolimits_{i \in I_\Omega} X_i, & X_W &:= \min\nolimits_{i \in I_\Omega} X_i, \\ Y_N &:= \max\nolimits_{i \in I_\Omega} Y_i, & Y_S &:= \min\nolimits_{i \in I_\Omega} Y_i \end{aligned}$$

denote the extremal coordinates of Ω. To visualize different directions we use N for the north-direction, E, S and W for east, south and west direction, respectively. Clearly, directions N, E, S and W can be understood as *top, right, bottom* or *left* direction, respectively. Without loss of generality, we may assume $X_W = 0$ and $Y_S = 0$. Consequently, target set Ω is completely contained within a rectangle R_0 of dimensions $L_0 = X_E - X_W$ and $W_0 = Y_N - Y_S$. Since all input-data are assumed to be integral we can restrict the coordinates of all allocation point to be integral too. Let $L_i = \{0, \dots, L_0 - \ell_i\}$, $W_i = \{0, \dots, W_0 - w_i\}$, $i \in I_n$. If rectangle R_i is placed with its lower left corner at point (p, q) with $p \in L_i$ and $q \in W_i$ then it covers the point set

$$\begin{aligned} R_i(p,q) &= \{(x,y) : p \le x < x + \ell_i, \\ & \qquad \le y < q + w_i\} \end{aligned} \tag{1}$$

where $\ell_i = 2a_i$ and $w_i = 2b_i$, $i \in I_n$. Notice, here the upper and right boundary are assumed not to be in $R_i(p,q)$ Let $\overline{\Omega}$ denote the minimal orthogonally convex rectilinear polygon with only integral corner points enveloping Ω. Since we are looking for a covering of Ω with only rectangles we have, the convex target set is covered if $\overline{\Omega}$ is covered. Notice, using an appropriate scale the approximation of Ω by $\overline{\Omega}$ is tight enough to obtain an exact solution. We assume that $\overline{\Omega}$ is a closed point set. According to definition (1) not all integer lattice points in $\overline{\Omega}$ have to be covered namely those lying on the NE or SE boundary, accept the leftmost point lying on the SE boundary. Let $\widetilde{\Omega}$ denote the integer lattice points in $\overline{\Omega}$ which have to be covered by the rectangles.

Then the optimization problem can be formulated as follows:

Beasley-type model: ILP model 1

$$\sum_{i \in I_n} \sum_{(p,q) \in \widetilde{\Omega}} c_i \, x_{ipq} \to \min \tag{2}$$

subject to

$$\sum_{i \in I_n} \sum_{p=s(i)}^{s} \sum_{q=t(i)}^{t} x_{ipq} \geq 1, \quad \forall \, (s,t) \in \widetilde{\Omega}, \tag{3}$$

$$\sum_{p \in L_i} \sum_{q \in W_i} x_{ipq} \leq 1, \quad \forall \, i \in I_n, \tag{4}$$

$$x_{ipq} \in \{0,1\}, \quad p \in L_i, \; q \in W_i, \; i \in I_n \tag{5}$$

where $s(i) = \max\{0, s - \ell_i + 1\}$, $t(i) = \max\{0, t - w_i + 1\}$.

This Beasley-type model has some drawbacks. First of all, the number of 0/1-variables depends on the dimensions of the target set. Moreover, the choice of an appropriate scale can further increase this number. Furthermore, the continuous (or linear programming, LP) relaxation (restrictions $x_{ipq} \in \{0,1\}$ are replaced by $x_{ipq} \in [0,1]$) is weak. It yields feasibility if the total area of the rectangles is not smaller than the area of $\overline{\Omega}$. This makes it difficult to solve the (integer) Beasley-type model.

Notice, the number of 0/1-variables can be somewhat reduced regarding the shape of $\overline{\Omega}$ instead of the enveloping rectangle $L \times W$. Another opportunity results from the principle of normalized patterns or raster points.

Basic Model

In the following, an attempt is made to model the CPR problem using a polynomial number of variables and restrictions.

Notations

The following sets of directions will be used:

$$D := \{N, E, S, W\},$$

$$D_t := \begin{cases} \{N, S\}, & \text{if } t \in \{E, W\}, \\ \{E, W\}, & \text{if } t \in \{N, S\}. \end{cases} \qquad t \in D.$$

Since the polygonal region Ω is assumed to be convex the following denotations and abbreviations are meaningful. We identify Ω defining functions visible from different directions as follows:

$$I_\Omega^{NW} := \{j \in I_\Omega : \frac{\delta g_j}{\delta x} < 0, \; \frac{\delta g_j}{\delta y} > 0\},$$

$$m_{NW} := |I_\Omega^{NW}|,$$

$$I_\Omega^{NE} := \{j \in I_\Omega : \frac{\delta g_j}{\delta x} > 0, \; \frac{\delta g_j}{\delta y} > 0\},$$

$$m_{NE} := |I_\Omega^{NE}|,$$

$$I_\Omega^{SW} := \{j \in I_\Omega : \frac{\delta g_j}{\delta x} < 0, \; \frac{\delta g_j}{\delta y} < 0\},$$

$$m_{SW} := |I_\Omega^{SW}|,$$

$$I_\Omega^{SE} := \{j \in I_\Omega : \frac{\delta g_j}{\delta x} > 0, \; \frac{\delta g_j}{\delta y} < 0\},$$

$$m_{SE} := |I_\Omega^{SE}|,$$

Moreover, let

$$m_{rs} = m_{sr} \quad \forall r \in D_s, \; s \in D.$$

In order to characterize the relation between two rectangles we define the constants

$$\begin{aligned} a_{ij} &:= \begin{cases} 1, & a_i > a_j, \\ 0, & a_i \leq a_j, \end{cases} \\ b_{ij} &:= \begin{cases} 1, & b_i > b_j, \\ 0, & b_i \leq b_j, \end{cases} \end{aligned} \qquad i, j \in I_n, \; i \neq j. \tag{6}$$

These constants are used to combine different cases, and therefore, to shorten the description.

For a given 0/1-vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ and a given vector $u = (u_1, \ldots, u_n) \in I\!R^{2n}$ of placement parameters $u_i \in I\!R^2$, $i \in I_n$ let

$$P(u, \alpha) := \bigcup_{i : \alpha_i = 1} R_i(u_i),$$

$$H(u, \alpha) := I\!R^2 \setminus \operatorname{int}(P(u, \alpha)).$$

Polygonal set $P(u, \alpha)$ represents the point set covered by the chosen rectangles whereas $H(u, \alpha)$ denotes the closure of the complement of $P(u, \alpha)$. For technical purposes let

$$d_t := \begin{cases} 4, & \text{if } t \in \{N, W\}, \\ 2, & \text{if } t \in \{S, E\}, \end{cases} \qquad t \in D. \tag{7}$$

Placement Parameters

As already introduced above, we denote the placement parameters (translation vectors) to be found by $u_i = (x_i, y_i)$, $i \in I_n$. Hence,

$$R_i(x_i, y_i) = \{(x, y) : \\ x_i - a_i \le x \le x_i + a_i, y_i - b_i \le y \le y_i + b_i\}$$

represents the region covered by the translated rectangle R_i.

Selection Variables

In order to indicate those rectangles R_i, $i \in I_n$ which are used to cover the polygonal region Ω we define 0/1-variables α_i according to

$$\alpha_i = \begin{cases} 1, & R_i(u_i) \text{ is used for the cover,} \\ 0, & R_i(u_i) \text{ is not used for the cover.} \end{cases}$$

Let R_i be given in the form

$$R_i(x_i, y_i) = \{(x, y) : f_i^r(x, y) \ge 0,\ r \in D\}, \\ i \in I_n,$$

where

$$f_i^N(x, y) := y_i + b_i - y,$$
$$f_i^E(x, y) := x_i + a_i - x,$$
$$f_i^S(x, y) := y + b_i - y_i,$$
$$f_i^W(x, y) := x + a_i - x_i.$$

In case that Ω fits within a single rectangle $R_i(u_i)$ and in related situations the following conditions on the translation vector $u_i = (x_i, y_i)$ are probably non-trivial since they form the boundary of $H(u, \alpha)$:

$$\widetilde{f}_i^N(x_i, y_i) := y_i + b_i - Y_N \ge 0,$$
$$\widetilde{f}_i^E(x_i, y_i) := x_i + a_i - X_E \ge 0,$$
$$\widetilde{f}_i^S(x_i, y_i) := Y_S + b_i - y_i \ge 0,$$
$$\widetilde{f}_i^W(x_i, y_i) := X_W + a_i - x_i \ge 0.$$

These inequalities should hold only if rectangle R_i is used. Therefore we modify them by adding some term depending on α_i:

$$\widetilde{f}_i^r(x_i, y_i) + M(1 - \alpha_i) \ge 0, \\ r \in D,\ i \in I_n, \tag{8}$$

where M is a sufficient large number, e. g. $M = \max\{L_0, W_0\}$. For every $i \in I_n$ these four restrictions (in a somewhat modified manner) have to be added to an ILP model.

If more then one rectangle is used to cover Ω not all of these restrictions have to be fulfilled depending on the relative positions of the rectangles. Here we assume that no coverings are of interest where some covering rectangle $R_i(u_i)$ is completely contained within another covering rectangle $R_j(u_j)$. Such a configuration cannot be optimal because of assumption $c_i > 0$, $i \in I_n$.

Relative Position Variables

If rectangles R_i and R_j are both used to cover Ω, i. e. if $\alpha_i = \alpha_j = 1$, 0/1-variables

$$\phi_{ij}^r \text{ and } \psi_{ij}^r, \quad i, j \in I_n,\ i \ne j,\ r \in \{1, \dots, 5\}$$

are introduced to characterize the relative position of the two rectangles to each other. In doing so we consider five different situations in horizontal and ialso n vertical direction (cf. Fig. **??**). Consequently, using both rectangles R_i and R_j that means we have

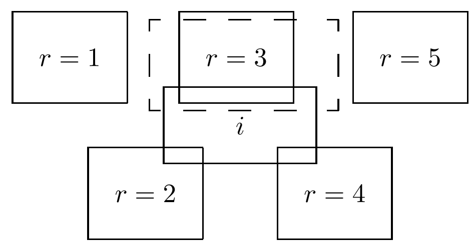

Figure 1: labelfig-1 Different relative horizontal positions

$$\sum_{r=1}^{5} \phi_{ij}^r = 1 \quad \text{and} \quad \sum_{r=1}^{5} \psi_{ij}^r = 1. \tag{9}$$

With other words, equations (9) should be fulfilled if and only if both rectangles R_i and R_j are used to cover Ω in order to get a unique description of the different configurations/interactions of the two rectangles which are as follows.

We define $\phi_{ij}^1 = 1$ if and only if $R_j(u_j)$ is completely left to $R_i(u_i)$ which leads to the inequality

$$x_j + a_j \le x_i - a_i + M(1 - \phi_{ij}^1) \\ \text{(non-trivial if } \phi_{ij}^1 = 1)$$

where M is a sufficient large number, e. g. $M = X_E + a_i + a_j$. The case when $R_j(u_j)$ is completely right to $R_i(u_i)$ is characterized by $\phi_{ij}^5 = 1$. If $\mathrm{int}(R_i(u_i)) \cap R_j(u_j) \ne \emptyset$ then we have three subcases, namely

$r = 2$: $x_j - a_j \le x_i - a_i \le x_j + a_j$,

$r = 3$: either $x_j - a_j \le x_i - a_i < x_i + a_i \le x_j + a_j$ or $x_i - a_i \le x_j - a_j < x_j + a_j \le x_i + a_i$,

$r = 4$: $x_j - a_j \le x_i + a_i \le x_j + a_j$.

Obviously, only one of two cases with $r = 3$ can occur according to the definition of a_{ij} and b_{ij} in formula (6). Altogether, we have the following restrictions (describing the relative horizontal position of R_i and R_j) which are non-trivial if some ϕ-variable has value one:

$$\begin{aligned}
&h^1_{ij}(x,\phi) := \\
&\quad x_i - a_i - x_j - a_j + M(1-\phi^1_{ij}) \ge 0, \\
&h^2_{ij}(x,\phi) := \\
&\quad x_j + a_j - x_i + a_i + M(1-\phi^2_{ij}) \ge 0, \\
&h^3_{ij}(x,\phi) := \\
&\quad x_i - a_i - x_j + a_j + M(1-\phi^2_{ij}) \ge 0, \\
&h^4_{ij}(x,\phi) := (a_{ij} - a_{ji})(x_j - a_j - x_i + a_i) \\
&\quad +M(1-\phi^3_{ij}) \ge 0, \\
&h^5_{ij}(x,\phi) := (a_{ij} - a_{ji})(x_i + a_i - x_j - a_j) \\
&\quad +M(1-\phi^3_{ij}) \ge 0, \\
&h^6_{ij}(x,\phi) := \\
&\quad x_j + a_j - x_i - a_i + M(1-\phi^4_{ij}) \ge 0, \\
&h^7_{ij}(x,\phi) := \\
&\quad x_i + a_i - x_j + a_j + M(1-\phi^4_{ij}) \ge 0, \\
&h^8_{ij}(x,\phi) := \\
&\quad x_j - a_j - x_i - a_i + M(1-\phi^5_{ij} - \ge 0.
\end{aligned} \tag{10}$$

In a similar way the ψ-variables are defined to characterize the relative vertical position of the two rectangles:

$$\begin{aligned}
&\widetilde{h}^1_{ij}(y,\psi) := \\
&\quad y_i - b_i - y_j - b_j + M(1-\psi^1_{ij}) \ge 0, \\
&\widetilde{h}^2_{ij}(y,\psi) := \\
&\quad y_j + b_j - y_i + b_i + M(1-\psi^2_{ij}) \ge 0, \\
&\widetilde{h}^3_{ij}(y,\psi) := \\
&\quad y_i - b_i - y_j + b_j + M(1-\psi^2_{ij}) \ge 0, \\
&\widetilde{h}^4_{ij}(y,\psi) := (b_{ij} - b_{ji})(y_j - b_j - y_i + b_i) \\
&\quad +M(1-\psi^3_{ij}) \ge 0, \\
&\widetilde{h}^5_{ij}(y,\psi) := (b_{ij} - b_{ji})(y_i + b_i - y_j - b_j) \\
&\quad +M(1-\psi^3_{ij}) \ge 0, \\
&\widetilde{h}^6_{ij}(y,\psi) := \\
&\quad y_j + b_j - y_i - b_i + M(1-\psi^4_{ij}) \ge 0, \\
&\widetilde{h}^7_{ij}(y,\psi) := \\
&\quad y_i + b_i - y_j + b_j + M(1-\psi^4_{ij}) \ge 0, \\
&\widetilde{h}^8_{ij}(y,\psi) := \\
&\quad y_j - b_j - y_i - b_i + M(1-\psi^5_{ij}) \ge 0.
\end{aligned} \tag{11}$$

Because of definition we have

$$\phi^r_{ij} = \phi^{6-r}_{ji},\ \psi^r_{ij} = \psi^{6-r}_{ji}, \quad r \in \{1,\ldots,5\},\ i,j \in I_n,\ i \ne j.$$

Notice, every combination of ϕ- and ψ-values which fulfills conditions (9) defines a subset of $I\!R^4$ of possible placement parameters $u_i = (x_i, y_i)$ and $u_j = (x_j, y_j)$.

Overlap Characterizing Variables

If two rectangles R_i and R_j are used to cover Ω then two essential different situations have to be considered: is the intersection of the two rectangles empty or not. For that reason 0/1-variables β_{ij} can be introduced which gets value one if and only if the intersection of $R_i(u_i)$ and $R_j(u_j)$ is non-empty.

It is obvious, the β-variables are dependent on the ϕ- and ψ-variables. It holds

$$\beta_{ij} = \sum_{r=2}^{4} \phi^r_{ij} \cdot \sum_{r=2}^{4} \psi^r_{ij}, \quad i,j \in I_n,\ i \ne j.$$

Note that although in cases $r = 1$ or $r = 5$ some "touching" is allowed, these situations cannot lead to a real overlap.

Inner Corners

Every pair R_i and R_j of used rectangles with non-empty intersection (i. e. $\beta_{ij} = 1$) determines some points, called *inner corners* which probably define a cone usable in the representation of $H(u,\alpha)$

In order to obtain a description of $H(u,\alpha)$ we consider all eight possibilities of inner corners which can arise. By means of 0/1-variables we identify in dependence of the ϕ- and ψ-variables these inner corners which are formed with respect to this configuration. Using further 0/1-variables we characterize those inner corners which form cones for the description of $H(u,\alpha)$.

In total, there are eight different types of inner corners. We define

$$E^{st}_{ij}, \quad s \in D,\ t \in D_s,\ i,j \in I_n,\ i \ne j,$$

as follows:

$$\begin{aligned}
E^{NW}_{ij} &:= (x_j - a_j, y_i + b_i), & E^{NE}_{ij} &:= (x_j + a_j, y_i + b_i), \\
E^{SW}_{ij} &:= (x_j - a_j, y_i - b_i), & E^{SE}_{ij} &:= (x_j + a_j, y_i - b_i), \\
E^{EN}_{ij} &:= (x_i + a_i, y_j + b_j), & E^{ES}_{ij} &:= (x_i + a_i, y_j - b_j), \\
E^{WN}_{ij} &:= (x_i - a_i, y_j + b_j), & E^{WS}_{ij} &:= (x_i - a_i, y_j - b_j).
\end{aligned}$$

In order to identify those inner corners which are induced by R_i, R_j, and the ϕ- and ψ-variables we define 0/1-variables ε_{ij}^{st} as follows.

The variable ε_{ij}^{NW} which corresponds to point $E_{ij}^{NW} = (x_j - a_j, y_i + b_i)$ has to be one if and only if

$$\alpha_i = \alpha_j = 1, \quad E_{ij}^{NW} \in R_i(u_i) \cap R_j(u_j).$$

In this case, the set

$$\{(x, y) : x \leq x_j - a_j,\ y \geq y_i + b_i\}$$

forms a cone in North-West direction. There are several situations where point E_{ij}^{NW} forms an inner corner, namely:

1. $\phi_{ij}^4 = 1$, $\psi_{ij}^4 = 1$,
2. $\phi_{ij}^4 = 1$, $\psi_{ij}^3 = 1$, $b_{ji} = 1$,
3. $\phi_{ij}^3 = 1$, $a_{ij} = 1$, $\psi_{ij}^4 = 1$,
4. $\phi_{ij}^3 = 1$, $a_{ij} = 1$, $\psi_{ij}^3 = 1$, $b_{ji} = 1$.

These four situations are depicted in Ffigure 2.

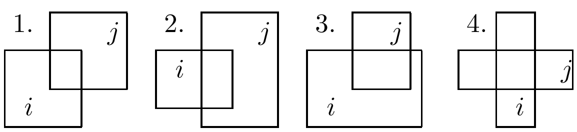

Figure 2: Situations where inner corner E_{ij}^{NW} can arise

Linear equations or inequalities are needed which ensure that ε_{ij}^{NW} becomes one in exactly these four cases. It holds for $i \neq j$:

$$\varepsilon_{ij}^{NW} = (\phi_{ij}^4 + a_{ij}\phi_{ij}^3)(\psi_{ij}^4 + b_{ji}\psi_{ij}^3),$$
$$\varepsilon_{ij}^{NE} = (\phi_{ij}^2 + a_{ij}\phi_{ij}^3)(\psi_{ij}^4 + b_{ji}\psi_{ij}^3),$$
$$\varepsilon_{ij}^{SW} = (\phi_{ij}^4 + a_{ij}\phi_{ij}^3)(\psi_{ij}^2 + b_{ji}\psi_{ij}^3),$$
$$\varepsilon_{ij}^{SE} = (\phi_{ij}^2 + a_{ij}\phi_{ij}^3)(\psi_{ij}^2 + b_{ji}\psi_{ij}^3).$$

Furthermore, for $i \neq j$ we have

$$\varepsilon_{ij}^{rs} = \varepsilon_{ji}^{sr}, \quad r \in D,\ s \in D_r.$$

If we suppose that $\varepsilon_{ij}^{NW} = 1$ and that $E_{ij}^{NW} \in H(u, \alpha)$ then at least one of the following inequalities, illustrated in Fig. 3,

must be fulfilled for the convex region Ω to ensure non-overlapping:

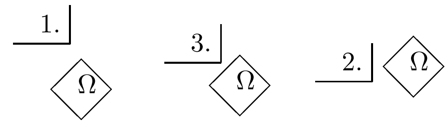

Figure 3: Interaction between E_{ij}^{NW} and target region Ω

1. Either $\Omega \subset \{(x, y) : y \leq y_i + b_i\}$, or
2. $\Omega \subset \{(x, y) : x \geq x_j - a_j\}$, or
3. $g_l(E_{ij}^{NW}) \geq 0$ for at least one $l \in I_\Omega^{NW}$.

This can be modelled using the ε-variables as follows::

$$\begin{aligned} \max\{&y_i + b_i - Y_N, X_W - x_j + a_j,\\ &\max\{g_l(E_{ij}^{NW}) : l \in I_\Omega^{NW}\}\}\\ &+M(1 - \varepsilon_{ij}^{NW}) \geq 0. \end{aligned} \tag{12}$$

For every pair of rectangles and every kind of potential inner corner such an inequality has to be considered, i. e. approximately $4n^2$ restrictions. But this inequality should be redundant if the point E_{ij}^{NW} is covered by a third rectangle.

Similar conditions hold for the other types of inner corners.

If only two rectangles are needed to cover Ω, i. e. $\sum_{i \in I_n} \alpha_i = 2$ and $\beta_{ij} = 1$, then the resulting two or four inner corners determine cones usable in the description of $H(u, \alpha)$.

If more than two rectangles are needed to cover Ω then some of the resulting inner corners can be covered by another third rectangle, and hence, are not useful for the description of $H(u, \alpha)$.

Active Inner Corners

In case of more than two rectangles used to cover Ω it may happen that some inner corner, e. g. E_{ij}^{NW}, is covered by a third rectangle R_k so that E_{ij}^{NW} does not form a part of the complement of the union of covering rectangles as drawn in Fig. 4. In order to charac-

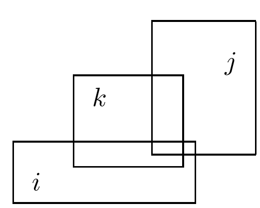

Figure 4: Inner corner E_{ij}^{NW} is not active

terize such situation we introduce a 0/1-variable ρ_{ij}^{NW} which has to be one if and only if E_{ij}^{NW} is not covered by another rectangle. More precisely, $\rho_{ij}^{NW}=1$ should hold true if and only if there does not exist any $k \in I_n \setminus \{i,j\}$ with

1. $\phi_{kj}^4 + a_{kj}\phi_{kj}^3 = 1$, and
2. $\psi_{ik}^4 + b_{ki}\psi_{ik}^3 = 1$.

Hence, we define

$$\rho_{ij}^{NW} = 1 \iff \sum_{k\neq i,j}(\phi_{kj}^4 + a_{kj}\phi_{kj}^3)(\psi_{ik}^4 + b_{ki}\psi_{ik}^3) = 0.$$

In case of $\rho_{ij}^{NW} = 1$ the bounding constraints in North-direction for R_i and in West-direction for R_j (according to (1)) must be removed. This can be achieved as follows:

$$\begin{aligned} &\widetilde{f}_i^N(x_i,y_i) + M(1-\alpha_i) \\ &\quad +M\textstyle\sum_{j:j\neq i}\rho_{ij}^{NW} \geq 0, \\ &\widetilde{f}_j^W(x_j,y_j) + M(1-\alpha_j) \\ &\quad +M\textstyle\sum_{j:j\neq i}\rho_{ij}^{NW} \geq 0. \end{aligned} \tag{13}$$

Instead of the two removed restrictions, another condition has to become relevant which guaranties that Ω is either in one of the two half-planes determined by f_i^N or f_j^W or that $E_{ij}^{NW} \notin \mathrm{int}(\Omega)$:

$$\begin{aligned} &\max\{\widetilde{f}_i^N(x_i,y_i), \widetilde{f}_j^W(x_j,y_j), \\ &\quad \max\{g_l(E_{ij}^{NW}) : l \in I_\Omega^{NW}\}\} \\ &\quad +M(1-\rho_{ij}^{NW}) \geq 0. \end{aligned}$$

Similar restrictions have to be introduced for the other types of active inner corners:

$$\begin{aligned} &\max\{\widetilde{f}_i^r(x_i,y_i), \widetilde{f}_j^s(x_j,y_j), \\ &\quad \max\{g_l(E_{ij}^{rs}) : l \in I_\Omega^{rs}\}\} \\ &\quad +M(1-\rho_{ij}^{rs}) \geq 0, \\ &\quad r\in D,\ s\in D_r,\ i,j\in I_n,\ i\neq j. \end{aligned} \tag{14}$$

Constraint Selection Variables

In case of an active inner corner E_{ij}^{rs}, i. e. with $\rho_{ij}^{rs} = 1$, 0/1-variables λ_{ijt}^{rs}, $t = 0,\ldots,m_{rs}+1$, are needed to identify a single constraint which ensures the non-overlapping similar to (12). Because of

$$\sum_{t=0}^{m_{rs}+1} \lambda_{ijt}^{rs} = \rho_{ij}^{rs}, \quad \forall r,s,i,j. \tag{15}$$

restriction (14) can be formulated as

$$\begin{aligned} &\lambda_{ij0}^{rs}\,\widetilde{f}_i^r(x_i,y_i) + \textstyle\sum_{t=1}^{m_{st}}\lambda_{ijt}^{rs}\,g_t^{rs}(E_{ij}^{rs}) \\ &\quad +\lambda_{ij,m_{rs}+1}^{rs}\,\widetilde{f}_j^s(x_j,y_j) \geq 0, \\ &\quad r\in D,\ s\in D_r,\ i,j\in I_n,\ i\neq j, \end{aligned} \tag{16}$$

where g_t^{rs} are the functions corresponding to I_Ω^{rs}.

Summation

If for all relations between the α-, ϕ-, ψ-, ρ-, mad λ- variables linear inequalities or equalities can be found then an ILP model for the problem of covering a convex polygon by rectangles can be obtained. As an intermediate result we have the following formulation of the CPR problem:

Compute placement parameters $x_i,\ y_i,\ i\in I_n$ and 0/1-variables α_i, $i \in I_n$, ϕ_{ij}^p, ψ_{ij}^p $p\in\{1,\ldots,5\}$, ρ_{ij}^{rs}, and λ_{ijt}^{rs}, $r\in D$, $s\in D_r$, $t\in\{0,\ldots,m_{rs}+1\}$, $i,j\in I_n$, $i\neq j$ such that

$$\sum_{i\in I_n} c_i\alpha_i \to \min, \tag{17}$$

subject to

$$\sum_{r=1}^{5}\phi_{ij}^r = \alpha_i\alpha_j, \quad \sum_{r=1}^{5}\psi_{ij}^r = \alpha_i\alpha_j, \quad \forall i,j, \tag{18}$$

$$h_{ij}^t(x,\phi) \geq 0, \quad t\in\{1,\ldots,8\},\ \forall i\neq j, \tag{19}$$

$$\widetilde{h}_{ij}^t(y,\psi) \geq 0, \quad t\in\{1,\ldots,8\},\ \forall i\neq j, \tag{20}$$

$$\begin{aligned} &\varepsilon_{ij}^{rs} = (\phi_{ij}^{d_s} + a_{ij}\phi_{ij}^3)(\psi_{ij}^{d_r} + b_{ji}\psi_{ij}^3), \\ &\quad r\in D,\ s\in D_r,\ i\neq j. \end{aligned} \tag{21}$$

$$\varepsilon_{ij}^{rs} = \varepsilon_{ji}^{sr}, \quad r\in D,\ s\in D_r. \tag{22}$$

$$\rho_{ij}^{rs} \leq \varepsilon_{ij}^{rs}, \quad \forall r\in D,\ s\in D_r,\ i\neq j, \tag{23}$$

$$\begin{aligned} &\rho_{ij}^{rs} = \varepsilon_{ij}^{rs} \ \Leftrightarrow\ \textstyle\sum_{k\neq i,j}(\phi_{kj}^{d_s} \\ &\quad +a_{kj}\phi_{kj}^3)(\psi_{ik}^{d_r} + b_{ki}\psi_{ik}^3) = 0, \\ &\qquad \forall r\in D,\ s\in D_r,\ i\neq j, \end{aligned} \tag{24}$$

$$\begin{aligned} &\widetilde{f}_i^r(x_i,y_i) + M(1-\alpha_i) \\ &\quad +M\textstyle\sum_{j\neq i}\sum_{s\in D_r}\rho_{ij}^{rs} \geq 0, \\ &\qquad r\in D,\ i\neq j, \end{aligned} \tag{25}$$

$$\begin{aligned} &\textstyle\sum_{t=0}^{m_{rs}|+1}\lambda_{ijt}^{rs} \\ &\qquad = \rho_{ij}^{rs}, \quad \forall r,s,i,j. \end{aligned} \tag{26}$$

$$\begin{aligned} &\lambda_{ij0}^{rs}\widetilde{f}_i^r(x_i,y_i) + \textstyle\sum_{t=1}^{m_{st}}\lambda_{ijt}^{rs}g_t^{rs}(E_{ij}^{rs}) \\ &+\lambda_{ij,m_{rs}+1}^{rs}\widetilde{f}_j^s(x_j,y_j) \geq 0, \\ &\quad r\in D,\ s\in D_r,\ i,j\in I_n,\ i\neq j. \end{aligned} \tag{27}$$

In the following we are going to develop a corresponding ILP model.

ILP Model 2

In order to get an ILP formulation with polynomial number of variables and constraints, all conditions in basic model (17) – (27) have to be formulated as linear restrictions.

Restricting the Placement Parameters

In order to ensure that R_i does not overlap Ω if R_i is not used, i. e. if $\alpha_i = 0$, we define the following inequalities:

$$\begin{aligned} &-a_i \le x_i \le (X_E + 2a_i)\alpha_i - a_i, \\ &-b_i \le y_i \le (Y_N + 2b_i)\alpha_i - b_i, \quad i \in I_n. \end{aligned} \tag{28}$$

Relations Between α-, ϕ- and ψ-Variables

Lower bounds for ϕ- and ψ-variables:

$$\begin{aligned} &0 \le \phi_{ij}^r,\ 0 \le \psi_{ij}^r, \\ &\quad \forall i, j \in I_n,\ i \ne j,\ r = 1, \dots, 5. \end{aligned} \tag{29}$$

Symmetry conditions (source of reducing the number of variables):

$$\begin{aligned} &\phi_{ij}^r = \phi_{ji}^{6-r},\ \psi_{ij}^r = \psi_{ji}^{6-r}, \\ &\quad r \in \{1, \dots, 5\},\ i, j \in I_n,\ i \ne j. \end{aligned} \tag{30}$$

Realization of the logical AND:

$$\begin{aligned} &\textstyle\sum_{r=1}^5 \phi_{ij}^r \le \alpha_i, \quad \sum_{r=1}^5 \phi_{ij}^r \le \alpha_j, \\ &\quad \textstyle\sum_{r=1}^5 \phi_{ij}^r \ge \alpha_i + \alpha_j - 1, \quad i, j \in I_n,\ i \ne j. \end{aligned} \tag{31}$$

$$\begin{aligned} &\textstyle\sum_{r=1}^5 \psi_{ij}^r \le \alpha_i, \quad \sum_{r=1}^5 \psi_{ij}^r \le \alpha_j, \\ &\quad \textstyle\sum_{r=1}^5 \psi_{ij}^r \ge \alpha_i + \alpha_j - 1, \quad i, j \in I_n,\ i \ne j. \end{aligned} \tag{32}$$

Moreover, inequalities (19) and (20) have to be fulfilled.

Relations Between β-, ϕ- and ψ-Variables

According to basic model (17) – (27) where no β-variables are used the following inequalities are not needed. On the other hand, if it is intended to exploit β-variables then these inequalities yield the relations between β-, ϕ- and ψ-variables.

Realization of the logical AND:

$$0 \le \beta_{ij},\ \forall i, j \in I_n,\ i \ne j,$$

$$\beta_{ij} \le \sum_{r=2}^4 \phi_{ij}^r,\ \beta_{ij} \le \sum_{r=2}^4 \psi_{ij}^r,$$

$$\beta_{ij} \ge \sum_{r=2}^4 \phi_{ij}^r + \sum_{r=2}^4 \psi_{ij}^r - 1.$$

Relations Between ε-, ϕ- and ψ-Variables

Naturally, we have

$$0 \le \varepsilon_{ij}^{st}, \quad \forall s \in D,\ t \in D_s,\ i, j \in I_n,\ i \ne j, \tag{33}$$

Realization of the logical AND:

$$\begin{aligned} &\varepsilon_{ij}^{rs} \le \phi_{ij}^{d_s} + a_{ij}\phi_{ij}^3, \quad \varepsilon_{ij}^{rs} \le \psi_{ij}^{d_r} + b_{ji}\psi_{ij}^3, \\ &\quad r \in D,\ s \in D_r,\ i \ne j, \end{aligned} \tag{34}$$

$$\begin{aligned} &\varepsilon_{ij}^{rs} \ge \phi_{ij}^{d_s} + a_{ij}\phi_{ij}^3 + \psi_{ij}^{d_r} + b_{ji}\psi_{ij}^3 - 1, \\ &\quad r \in D,\ s \in D_r,\ i \ne j, \end{aligned} \tag{35}$$

Because of definition:

$$\varepsilon_{ij}^{rs} = \varepsilon_{ji}^{sr} \quad \forall i, j, r, s. \tag{36}$$

Relations Between ε- and ρ-Variables

By definition we have

$$0 \le \rho_{ij}^{rs} \le \varepsilon_{ij}^{rs}, \quad \forall r \in D,\ s \in D_r,\ i \ne j, \tag{37}$$

In order to get an ILP formulation for (24), i. e. for

$$\begin{aligned} &\rho_{ij}^{rs} = \varepsilon_{ij}^{rs} \ \Leftrightarrow \\ &\textstyle\sum_{k \ne i,j} (\phi_{kj}^{d_s} + a_{kj}\phi_{kj}^3)(\psi_{ik}^{d_r} + b_{ki}\psi_{ik}^3) = 0, \\ &\quad \forall r \in D,\ s \in D_r,\ i \ne j, \end{aligned}$$

we introduce 0/1-variables ρ_{ijk}^{rs} by

$$\rho_{ijk}^{rs} = 1 \quad \Leftrightarrow \quad (\phi_{kj}^{d_s} + a_{kj}\phi_{kj}^3)(\psi_{ik}^{d_r} + b_{ki}\psi_{ik}^3) = 1,$$
$$\forall r \in D,\ s \in D_r,\ i \ne j,\ k \in I_n \setminus \{i, j\}.$$

This can be modelled as follows:

$$\begin{aligned} &0 \le \rho_{ijk}^{rs} \le \phi_{kj}^{d_s} + a_{kj}\phi_{kj}^3, \\ &\rho_{ijk}^{rs} \le \psi_{ik}^{d_r} + b_{ki}\psi_{ik}^3, \\ &\rho_{ijk}^{rs} \ge \phi_{kj}^{d_s} + a_{kj}\phi_{kj}^3 + \psi_{ik}^{d_r} + b_{ki}\psi_{ik}^3 - 1, \\ &\quad \forall r \in D,\ s \in D_r,\ i \ne j,\ k \in I_n \setminus \{i, j\}. \end{aligned} \tag{38}$$

Now we have the condition

$$\rho_{ij}^{rs} = 1 \quad \Leftrightarrow \quad \sum_{k \ne i,j} \rho_{ijk}^{rs} = 0$$

which is modelled as

$$\begin{aligned}&\varepsilon_{ij}^{rs}-\rho_{ij}^{rs}\leq\sum_{k\neq i,j}\rho_{ijk}^{rs},\\&(n-2)(\varepsilon_{ij}^{rs}-\rho_{ij}^{rs})\geq\sum_{k\neq i,j}\rho_{ijk}^{rs},\\&\qquad\forall r\in D,\ s\in D_r,\ i\neq j.\end{aligned}\tag{39}$$

In the last condition it is assumed that at least two rectangles are available to build a cover, i. e. $n \geq 2$.

Moreover, we have

$$\rho_{ij}^{rs}=\rho_{ji}^{sr},\quad\forall r,s,i,j.\tag{40}$$

and, last but not least,

$$\begin{aligned}&\widetilde{f}_i^r(x_i,y_i)+M(1-\alpha_i)\\&\quad+M\sum_{j\neq i}\sum_{s\in D_r}\rho_{ij}^{rs}\geq 0,\quad r\in D,\ i\neq j,\end{aligned}\tag{41}$$

$$\sum_{t=0}^{m_{rs}|+1}\lambda_{ijt}^{rs}=\rho_{ij}^{rs},\quad\forall r,s,i,j.\tag{42}$$

$$\begin{aligned}&\lambda_{ij0}^{rs}\widetilde{f}_i^r(x_i,y_i)+\sum_{t=1}^{m_{st}}\lambda_{ijt}^{rs}g_t^{rs}(E_{ij}^{rs})\\&\quad+\lambda_{ij,m_{rs}+1}^{rs}\widetilde{f}_j^s(x_j,y_j)\geq 0,\\&\quad r\in D,\ s\in D_r,\ i,j\in I_n,\ i\neq j.\end{aligned}\tag{43}$$

Feasibility

In this model, objective function (17) and restrictions (28) – (43), it is possible that no covering exists. For computational purposes it may be better to have the existence of feasible solutions in the optimization problem.

There are several possibilities to guarantee feasible solutions, e. g. by adding artificial rectangles with sufficient high costs, or by weakening some of the restrictions similar to [9].

The Linear Model

Besides the linear objective function (17) and the linear restrictions (19), (20), (22), (23), (25) – (27) of the basic model now we have to add all linear constraints (28) – (43) to get a linear formulation of the CPR problem with continuous and binary variables. In this formulation we have
$2n$ continuous variables x_i, y_i, $i \in I_n$,
n rectangle selection variables α_i, $i \in I_n$,
$10n(n-1)$ relative position variables ϕ_{ij}^r and ψ_{ij}^r, $i \neq j \in I_n$, $r \in \{1, \dots, 5\}$,
$8n(n-1)$ inner corner identification variables ϵ_{ij}^{rs}, $i \neq j \in I_n$, $r \in D$, $s \in D_r$,
$8n(n-1)$ active inner corner identification variables ρ_{ij}^{rs}, $i \neq j \in I_n$, $r \in D$, $s \in D_r$,
$O(n^2 m)$ constraint selection variables λ_{itj}^{rs}, $i \neq j \in I_n$, $r \in D$, $s \in D_r$, $t \in I_\Omega$, and
$O(n^3)$ variables ρ_{ijk}^{rs}, $i \neq j \neq k \in I_n$, $r \in D$, $s \in D_r$.
Hence, the total number of variables is proportional to $n^2 \cdot \max\{n, m\}$.

The number of constraints has the same order of magnitude.

Extensions

Here we propose directions of further research.

ILP Model 3: Non-convex Region Ω

If Ω is the union of a finite number of convex polygons, i. e.

$$\Omega=\bigcup_q\Omega_q\quad\text{ where }\Omega_q\text{ is convex for all }q,\tag{44}$$

then, in analogy to [9], for every subset Ω_q a complete set of λ-variables has to be introduced. The placement parameters x_i and y_i and the α-, ϕ- and ψ-variables are maintained.

An Alternative ILP Model

Another way of modelling is as follows. Given the ϕ- and ψ-values we can obtain the ε-values. For a certain subset Ω_q we derive the ρ-values regarding only these rectangles which are used to cover Ω_q. For every Ω_q a set of α-, ρ- and λ-variables is needed.

Restricting the Placement Parameters

For every $q \in Q$ 0/1-variables α_{iq} are defined. Then, a rectangle R_i, $i \in I_n$, is used to cover Ω if for at least one $q \in Q$ $\alpha_{iq} = 1$ holds true. In order to ensure that R_i does not overlap $\Omega = \bigcup_{q\in Q} \Omega_q$ if R_i is not used for any subset of Ω, i. e. if $\sum_{q\in Q} \alpha_{iq} = 0$, we define the following inequalities:

$$\begin{aligned}&-a_i\leq x_i\leq(X_E+2a_i)\sum_{q\in Q}\alpha_{iq}-a_i,\\&-b_i\leq y_i\leq(Y_N+2b_i)\sum_{q\in Q}\alpha_{iq}-b_i,\\&\quad i\in I_n.\end{aligned}\tag{45}$$

Conclusions

Two ILP models have been developed for problem CPR. In the first one, the number of variables and constraints is dependent on the size of the target region; in the second model it is polynomially bounded but even large. Investigations to reduce this number as well as numerical experiments are needed. Moreover, alternative formulations can possibly help.

References

[1] V. S. Anil Kumar and H. Ramesh. Covering rectlinear polygons with axis-parallel rectangles. SIAM J. Comput. vol. 32, no. 6, 2003, pp. 1509–1541.

[2] J. E. Beasley. Bounds for two-dimensional cutting. J. Oper. Res. Soc., vol. 36, no. 1, pp. 71–74, 1985.

[3] J. E. Beasley. An exact two-dimensional non-guillotine cutting tree search procedure. Oper. Res., vol. 33, no. 1, pp. 49–64, 1985.

[4] K. Daniels and R. Inkulu. An incremental algorithm for translational polygon covering. Technical Report 2001-001, University of Massachusetts at Lowell Computer Science, 2001.

[5] H. Dyckhoff, G. Scheithauer, and J. Terno. Cutting and packing. In M. Dell?Amico, F. Maffioli, and S. Martello, editors, Annotated Bibliographies in Combinatorial Optimization, chapter 22, pp. 393–412. John Wiley & Sons, Chichester, 1997.

[6] M. R. Garey and D. S. Johnson. Computers and Intractability – A Guide to the Theory of NP- Completeness. Freeman, San Francisco, 1979.

[7] A. Lingas, A. Wasylewicz, and P. Zylinski. Note on covering monotone orthogonal polygons with star-shaped polygons. Information Processing Letters vol. 104, no. 6, pp. 220–227, 2007.

[8] G. Scheithauer. One-dimensional relaxations for the problem of covering a polygonal region by rectangles. Preprint MATH-NM-04-2009, Technische Universit"at Dresden, 2009.

[9] G. Scheithauer, Yu. Stoyan, T. Romanova, and A. Krivulya. Covering a polygonal region by rectangles. Computational Optimization and Applications, Springer, online: http://www.citeulike.org/user/ima/article/4575123

[10] San-Yuan Wu and Sartaj Sahni. Covering rectlinear polygons by rectangles. Computer-Aided Design of Integrated Circuits and Systems, IEEE Transactions on. 01/05/1990; 9(4):377-388.

G. Scheithauer 1983 Doctor rer. nat. at Technische Universit"at Dresden, Germany, since 1983 member of the scientific staff at the Institute for Numerical Mathematics, Department of Mathematics, TU Dresden.
Selected papers: 1. Zuschnitt- und Packungsoptimierung, Vieweg + Teubner, 2008 (in German, 338 pages) 2. Gomory Cuts from a Position-Indexed Formulation of 1D Stock Cutting. In Bortfeldt et al (eds), Intelligent Decision Support, Gabler Edition Wiss. 2008, pp. 1-14 (with G. Belov, ,C. Alves, J.M.V. de Carvalho) 3. LP-based bounds for the Container and Multi-Container Loading Problem. Int. Trans. Opl. Res., 6 (1999) 199–213.
Address: Institute for Numerical Mathematics, Department of Mathematics, Technische Universit"at Dresden, 01062 Dresden, Germany,
Field of specialisation: Operational Research, Mathematical modeling, Optimization, Cutting, Packing, Covering
Member of ESICUP

Yu. Stoyan 1966 Ph. D./Candidat in Physical-Mathematical Sciences, Institute of Cybernetics of National Academy of Sciences of Ukraine, Kiev, 1970 Doctor of Technical Science, Aviation Engineering Institute in Moscow, 1972 Professor for Computational Mathematics, the Kharkov National University of Radioelectronics, 1985 Corresponding-member of National Academy of Sciences of Ukraine, since 1972 Head of Department of Mathematical Modeling at the Institute for Mechanical Engineering Problems of the National Academy of Sciences of Ukraine.
Selected papers: 1. Packing of convex polytopes into a parallelepiped, Optimization, vol. 54, 2005, pp. 215-235.(with N. Gil, G. Scheithauer, A. Pankratov, I. Magdalina) 2. Packing of Various Radii Solid Spheres into a Parallelepiped. Central Europen Journal of Operational Research, 2003, Vol. 11, Issue 4, pp.389-407. (with G. Yaskov, G. Scheithauer) 3. Packing cylinders and rectangular parallelepipeds with distances between them, Europ. J. Oper. Research 197, 2008, 446-455.(with A. Chugay).
Address: Institute for Problems in Machinery, National Ukrainian Academy of Sciences, 2/10 Pozharsky St., Kharkov, 61046, Ukraine
Field of specialisation: Computational Geometry, Operational Research, Mathematical modeling, Optimization, Cutting, Packing, Covering
Member of ESICUP

T. Romanova 1990 Ph. D./Candidat in Physical-Mathematical Sciences, Institute of Cybernetics of National Academy of Sciences of Ukraine, Kiev, 2003 Doctor of Technical Science, Institute of Cybernetics of National Academy of Sciences of Ukraine, Kiev, 2005 Professor for Computational Mathematics, the Kharkov National University of Radioelectronics, since 2003 Senior staff scientist of Department of Mathematical Modeling at the Institute for Mechanical Engineering Problems of the National Academy of Sciences of Ukraine.
Selected papers: 1. Phi-functions for complex 2D-objects//4OR Quarterly Journal of the Belgian, French and Italian Operations Research Societies. vol. 2, no 1, 2004 pp. 69 - 84. (with G. Scheithauer, N. Gil, Yu. Stoyan) 2. Construction of a Phi-function for two convex polytopes, Applicationes Mathematicae, 2002, Vol.29, No 2, pp. 199 - 218. (with Yu. Stoyan, J. Terno, M. Gil, G. Scheithauer) 3. Phi-function for 2D primary objects//Studia Informatica Universalis, 2002, Vol. 2, No 1, pp. 1-32. (with Yu. Stoyan, J. Terno, G. Scheithauer, N. Gil)
Address: Institute for Problems in Machinery, National Ukrainian Academy of Sciences, 2/10 Pozharsky St., Kharkov, 61046, Ukraine
Field of specialisation: Computational Geometry, Operational Research, Mathematical modeling, Optimization, Cutting, Packing, Covering
Member of ESICUP